Available online at www.isr-publications.com/mns
Math. Nat. Sci., 5 (2019), 29–43
Research Article

Online: ISSN 2600-7665
Mathematics in Natural Science
An International Journal
Journal Homepage: www.isr-publications.com/mns

A new result on the global exponential stability of nonlinear neutral volterra integro-differential equation with variable lags

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Abstract

In this study, the global exponential stability (GES) of the zero solution of a nonlinear neutral volterra integro-differential equation (NVIDE) with variable lags has been investigated. Based on the Lyapunov functional approach, a new stability criterion was derived for global exponential stability criterions of the considered equation. An example with numeric simulation has been given to demonstrate the applicability and accuracy of the obtained result by MATLAB Simulink.

Keywords: NVIDE, GES, Lyapunov functional, variable lags.

2010 MSC: 34K20, 34K40.

1. Introduction

The stability analysis of neutral differential equations/systems has been investigated extensively by many researchers for the past few decades. It is well known that this equations or systems are frequently used in various practical engineering systems such as power systems, aircraft, chemical and control systems. The question of asymptotic or exponential stability of neutral differential equations (NDEs) is very important both theoretically and practically. Recently, the stability analysis for neutral equations with time-lags has been examined by many authors.

It should be noted that, Agarwal and Grace [1] consider the NDE form

$$\frac{d}{dt}[x + c(t)x(t-\tau)] + p(t)x = q(t)\tanh x(t-\sigma), \qquad t \geqslant 0, \tag{1.1}$$

where τ and σ are positive real constants $\sigma \geqslant \tau, c, p, q : [t_0, \infty) \to [t_0, \infty)$ are continuous, and $c(t)$ is differentiable with locally bounded derivative. The authors obtained sufficient conditions under which all solutions of the NDE (1.1) approach zero as $t \to \infty$.

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doi: 10.22436/mns.05.01.04

Received: 2018-02-18 *Revised:* 2019-12-02 *Accepted:* 2019-12-06

At the same time, Keadnarmol and Rojsiraphisal [11] considered the NDE form

$$\frac{d}{dt}[x + px(t - \tau(t))] = -ax + b\tanh x(t - \sigma(t)). \tag{1.2}$$

Using the Lyapunov functional approach, the authors established some sufficient conditions for the GES of solutions of NDE (1.2).

In addition, Park and Kwon [18] presented that all solutions of NDE

$$\frac{d}{dt}[x + px(t - \tau)] + ax - b\tanh x(t - \sigma), \quad t \geqslant 0, \tag{1.3}$$

approach zero at $t \to \infty$.

In the related literature, the most of researchers have focused on the qualitative properties of special case of NDEs (1.1)–(1.3) with constant or variable coefficients and several lags. During the investigations, the authors benefited from different methods such as the Liapunovs function method, integral inequalities, linear matrix inequality, perturbation techniques, model transformations, etc. (see [1–23] and the references therein).

In this paper, instead of the NDEs motivated by the preceding discussion we consider the following nonlinear NVIDE with variable lags:

$$\begin{aligned}\frac{d}{dt}[x + \sum_{i=1}^{2} p_i(t)x(t - \tau_i(t))] = &- a(t)h(x) + \sum_{i=1}^{2} b_i(t)\tanh x(t - \tau_i(t)) \\ &+ \sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} c(t,s)g(x(s))ds, \quad t \geqslant 0,\end{aligned} \tag{1.4}$$

where $g, h : \mathbb{R} \to \mathbb{R}, c : \Phi \to \mathbb{R}, a, b$ and p are continuous functions on their respective domains, $\Phi := \{(t,s) : 0 \leqslant s < t < \infty\}$ with $g(0) = 0, h(0) = 0$; the function $b_i, i = (1,2)$ is continuous and differentiable, and $\left|\sum_{i=1}^{2} p_i(t)\right| \leqslant p_0 < 1$, ($p_0$-constant). The variable lags $\tau_i(t)$ are continuous and differentiable, which are defined by $\tau_i(t) : [0,\infty) \to [0,\tau_i]$ satisfying

$$0 \leqslant \tau_i(t) \leqslant \tau_i, \quad \tau_i'(t) \leqslant \delta_i < 1, \tag{1.5}$$

where $\tau_i > 0$ and $\delta_i > 0$ for $i = 1,2$.

Throughout this paper, $x(t)$ is abbreviated as x.

For NVIDE (1.4), we assume the existence of initial condition

$$x_0(\theta) = \phi(\theta), \quad \theta \epsilon [-\tau_i, 0],$$

where $\phi \epsilon C([-\tau_i, 0]; \mathbb{R}), \quad (i = 1,2)$.

Define

$$h_1(x) = \begin{cases} \frac{h(x)}{x}, x \neq 0, \\ \frac{dh(0)}{dt}, x = 0, \end{cases} \tag{1.6}$$

and

$$g_1(x) = \begin{cases} \frac{g(x)}{x}, x \neq 0, \\ \frac{dg(0)}{dt}, x = 0. \end{cases} \tag{1.7}$$

From (1.6) and (1.7), NVIDE (1.4) can be readily rewritten as follows:

$$\frac{d}{dt}[x+\sum_{i=1}^{2} p_i(t)x(t-\tau_i(t))] = -a(t)h_1(x)x+\sum_{i=1}^{2} b_i(t)\tanh x(t-\tau_i(t))$$
$$+\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} c(t,s)g_1(x(s))x(s)ds, \quad t \geqslant 0. \tag{1.8}$$

The main purpose and contribution of this study can be summarized as follows aspects:

(i) This research on the GES of NVIDE is still at the stage of exploiting and developing. Therefore, we propose a new stability criterion for further improvements.

(ii) The technique of proof in this study involves some basic inequalities and Lyapunov function method. The obtained stability criteria are defined as matrix inequalities which are also convenient and feasible to test the GES of the addressed nonlinear NVIDE.

(iii) Compared to the existing results in [1, 2, 5–7, 11, 12, 14, 16–21], the results of this study are more general. In addition, an example with numerical simulation is given to show the effectiveness of the obtained theoretical result. The result presented in this study contributes to the neutral type delayed equations and related ones.

2. Preliminaries and stability results

For convenience, let $D_1(t) = x+\sum\limits_{i=1}^{2} p_i(t)x(t-\tau_i(t))$. Thus, NVIDE (1.8) can be readily written as

$$D_1' = \frac{d}{dt}[x+\sum_{i=1}^{2} p_i(t)x(t-\tau_i(t))] = -a(t)h_1(x)x+\sum_{i=1}^{2} b_i(t)\tanh x(t-\tau_i(t))$$
$$+\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} c(t,s)g_1(x(s))x(s)ds, \quad t \geqslant 0.$$

With the new variable defined above, we can write the descriptor form of NVIDE (1.8) as follows:

$$\begin{cases} D_1' = -a(t)h_1(x)x+\sum\limits_{i=1}^{2} b_i(t)\tanh x(t-\tau_i(t))+\sum\limits_{i=1}^{2}\int_{t-\tau_i(t)}^{t} c(t,s)g_1(x(s))x(s)ds, \\ 0 = -D_1+x+\sum\limits_{i=1}^{2} p_i(t)x(t-\tau_i(t)). \end{cases} \tag{2.1}$$

We assume that there exist nonnegative $\sigma, a_i, m_i, n_i, p_{1i}, p_{2i}$ and positive constants b_{1i}, b_{2i} such that for $0 \leqslant s < t < \infty$ and $t \geqslant t_0$,

$$a_1 \leqslant a(t) \leqslant a_2, \quad c(t,s) \leqslant \sigma, \quad b_{1i} \leqslant b_i(t) \leqslant b_{2i}, \quad b_i'(t) \leqslant 0, \tag{2.2}$$

$$p_{1i} \leqslant p_i(t) \leqslant p_{2i}, \quad \left|\sum_{i=1}^{2} p_i(t)\right| \leqslant p_0 < 1, \quad m_1 \leqslant g_1(x) \leqslant m_2, \quad n_1 \leqslant h_1(x) \leqslant n_2, \tag{2.3}$$

for $i = 1, 2$.

Before state the main result, the following basic definition, lemma, proposition and theorem are needed.

Definition 2.1. The solution $0 = x$ of NVIDE (1.8) is exponential stability if

$$\|x\| \leqslant K\exp(-\lambda t)sub_{-\tau_i\leqslant s\leqslant 0}\|x(s)\| = K\exp(-\lambda t)\|x_0\|_s, \tag{2.4}$$

where $K > 0\epsilon R$, $\lambda(>0)\epsilon R$ and $\|x_t\|_s = sub_{-\tau_i\leqslant s\leqslant 0}\|x(t+s)\|$, $i = (1, 2)$.

Lemma 2.2. *Let* $N \epsilon R^{n\times n}$ *be any symmetric positive-definite matrix and* $x, y \epsilon R^n$. *Then*

$$\pm 2x^T y \leqslant x^T N x + y^T N^{-1} y.$$

Proposition 2.3. *Let* $M_i > 0, \mu_i > 0, \left|\sum_{i=1}^{2} p_i(t)\right| \leqslant p_0 < 1$ *and* $0 \leqslant \tau_i(t) \leqslant \tau_i, (i = 1, 2)$. *If* $x : [-\tau_i, \infty) \to R$ *satisfies*

$$\|x\| \leqslant \sup_{s\epsilon[-\tau_i,0]} \|x(s)\| = \|x_0\|_s, t\epsilon[-\tau_i, 0],$$

and

$$\|x\| \leqslant p_0\|x(t - \tau_i(t))\| + M_i exp(-\mu_i t),$$

then three are positive constants $\varepsilon_i, m_i \epsilon \left[0, \frac{-\ln p_0}{\tau_i}\right]$ *such that*

$$p_0 exp(\varepsilon\tau_i) < 1,$$

and

$$\|x\| \leqslant \|x_0\|_s exp(-m_i t)\| + \frac{M_i}{1 - p_0 exp(\varepsilon_i\tau_i)} exp(-\varepsilon_i t) \leqslant N_i exp(-\vartheta_i t), \tag{2.5}$$

where $t \geqslant 0, N_i = \|x_0\|_s + \frac{M_i}{1-p_0 exp(\varepsilon_i\tau_i)}$ *and* $\vartheta_i = \min\{m_i, \varepsilon_i\}$ *for* $i = 1, 2$.

Proof. Considering the assumptions $\left|\sum_{i=1}^{2} p_i(t)\right| \leqslant p_0 < 1$ and $0 \leqslant \tau_i(t) \leqslant \tau_i$, one can find sufficient small positive constant $\varepsilon_i, m_i \epsilon \left[0, \frac{-\ln p_0}{\tau_i}\right]$ such that $p_0 exp(\varepsilon_i\tau_i) < 1$ and $p_0 exp(m_i\tau_i) < 1, i = (1, 2)$. We verify that inequality (2.5) is true. If $\mu_i \leqslant \varepsilon_i$, we can choose $\mu_i = \varepsilon_i$, else if $\mu_i > \varepsilon_i$, we have $exp(-\mu_i t) \leqslant exp(\varepsilon_i t)$ for $i = 1, 2$.

For $t = 0$, we have

$$\begin{aligned}\|x(0)\| &\leqslant p_0\|x(-\tau_i(0))\| + M_i \leqslant p_0 \sup_{-\tau_i\leqslant s\leqslant 0}\|x(s)\| + M_i \\ &< \|x_0\|_s + \frac{M_i}{1 - p_0 exp(\varepsilon_i\tau_i)} \equiv N_i, (i = 1, 2).\end{aligned}$$

Therefore, the inequality (2.5) is true when $t = 0$.

Now, let $t > 0$. Suppose that inequality (2.5) fails. Hence, there is $t^* > 0$ such that

$$\|x(t^*)\| > \|x_0\|_s exp(-m_i t^*) + \frac{M_i}{1 - p_0 exp(\varepsilon_i\tau_i)} exp(-\varepsilon_i t^*) \equiv N_i exp(-\vartheta_i t), \tag{2.6}$$

and

$$\|x(t)\| \leqslant \|x_0\|_s exp(-m_i t) + \frac{M_i}{1 - p_0 exp(\varepsilon_i\tau_i)} exp(-\varepsilon_i t) \equiv N_i exp(-\vartheta_i t),$$

for all $t\epsilon[0, t^*)$ and $i = 1, 2$.

A1. Let $t^* > \tau_i(t^*) > 0, i = (1, 2)$. Then

$$\|x(t^*)\| \leqslant p_0\|x(t^* - \tau_i(t^*))\| + M_i exp(-\mu_i t^*)$$

$$
\begin{aligned}
&\leqslant p_0\{\|x_0\|_s \exp(-m_i(t^*-\tau_i(t^*))) + \frac{M_i}{1-p_0\exp(\varepsilon_i\tau_i)}\exp(-\varepsilon_i(t^*-\tau_i(t^*)))\} \\
&\quad + M_i\exp(-\varepsilon_i t^*) \\
&\leqslant p_0\exp(m_i\tau_i)\|x_0\|_s\exp(-m_i t^*) + \frac{M_i p_0 \exp(\varepsilon_i\tau_i)}{1-p_0\exp(\varepsilon_i\tau_i)}\exp(-\varepsilon_i t^*) \\
&\quad + M_i\exp(-\varepsilon_i t^*) \\
&\leqslant \|x_0\|_s\exp(-m_i(t^*) + \frac{M_i}{1-p_0\exp(\varepsilon_i\tau_i)}\exp(-\varepsilon_i t^*) \equiv N_i\exp(-\vartheta t^*).
\end{aligned}
$$

A2. Let $-\tau_i < 0 < t^* < \tau_i(t^*), (i = 1,2)$. Then

$$\|x(t^*-\tau_i(t^*))\| \leqslant \|x_0\|_s = \sup_{s\in[-\tau_i,0]}\|x(s)\|,$$

and hence, it follows that

$$
\begin{aligned}
\|x(t^*)\| &\leqslant p_0\|x(t^*-\tau_i(t^*))\| + M_i\exp(-\mu_i t^*) \\
&\leqslant \|x_0\|_s\exp(-m_i t^*) + \frac{M_i}{1-p_0\exp(\varepsilon_i\tau_i)}\exp(-\varepsilon_i t^*) \equiv N_i\exp(-\vartheta t^*),
\end{aligned}
$$

for $i = 1,2$.

It is clear that there is a contradiction to inequality (2.6) for both cases A1 and A2. Therefore, inequality (2.5) is true for all $t \geqslant 0$. □

Theorem 2.4. *Let $\sigma, a_i, m_i, n_i, (i = 1,2)$ be nonnegative constants. Then trivial solution of NVIDE (1.8) with (1.5) is GES if $\left|\sum_{i=1}^{2} p_i(t)\right| \leqslant p_0 < 1$ and there exist positive constants $b_{1i}, b_{2i}, \rho_1, \alpha, k$ and constants $\rho_j, (i = 1,2$ and $j = 2,3,\cdots,6)$, such that*

$$
\Xi = \begin{bmatrix}
2k\rho_1-2\rho_2 & \Xi_{12} & \Xi_{13} & \Xi_{14} & \rho_1 b_{21} & \rho_1 b_{22} & \Xi_{17} & \rho_3 \\
* & \Xi_{22} & 4\rho_6 & 4\rho_6 & 0 & 0 & \rho_4 & 4\rho_6 \\
* & * & \Xi_{33} & -2\rho_6 & 0 & 0 & \Xi_{37} & -2\rho_6 \\
* & * & * & \Xi_{44} & 0 & 0 & \Xi_{47} & -2\rho_6 \\
* & * & * & * & -b_{11}(1-\delta_1) & 0 & 0 & 0 \\
* & * & * & * & * & -b_{12}(1-\delta_2) & 0 & 0 \\
* & * & * & * & * & * & 2\rho_5 & \rho_5 \\
* & * & * & * & * & * & * & -2\rho_6
\end{bmatrix} < 0, \tag{2.7}
$$

where $\Xi_{12} = -\rho_1 a_1 n_1 + \rho_2 - 2\rho_3$, $\Xi_{13} = \rho_2 p_{21} + \rho_3$, $\Xi_{14} = \rho_2 p_{22} + \rho_3$, $\Xi_{17} = \sigma\rho_1 m_2 - \rho_4$, $\Xi_{22} = 2(-4\rho_6 + \rho_5) + (\alpha + b_{21})e^{2k\tau_1} + (\alpha + b_{22})e^{2k\tau_2}$, $\Xi_{33} = -\alpha(1-\delta_1) - 2\rho_6$, $\Xi_{37} = \rho_4 p_{21} + \rho_5$, $\Xi_{44} = -\alpha(1-\delta_2) - 2\rho_6$, $\Xi_{47} = \rho_4 p_{22} + \rho_5$ and the symbols " $*$ " shows the elements below the main diagonal of the symmetric matrix Ξ.

Proof. To prove the theorem, we define a new Lyapunov functional as follows

$$V(t) = V_1(t) + V_2(t) + V_3(t),$$

where

$$
\begin{aligned}
V_1(t) =& e^{2kt}[D_1, \sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} x(s)ds, 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \rho_1 & 0 & 0 \\ \rho_2 & \rho_3 & 0 \\ \rho_4 & \rho_5 & \rho_6 \end{bmatrix} \begin{bmatrix} D_1 \\ \sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} x(s)ds \\ 0 \end{bmatrix} \\
=& e^{2kt}\rho_1 D_1^2,
\end{aligned}
$$

$$
\begin{aligned}
V_2(t) =&\alpha\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} e^{2k(s+\tau_i)}x^2(s)ds+\sum_{i=1}^{2}b_i(t)\int_{t-\tau_i(t)}^{t} e^{2k(s+\tau_i)}\tan h^2 x(s)ds,\\
V_3(t) =&\eta e^{2kt}D_1^2,
\end{aligned}
$$

where $D_1 = x+\sum_{i=1}^{2}p_i(t)x(t-\tau_i(t)), \rho_1 > 0, \alpha > 0, b_i(t) > 0, \rho_j \in R, (i = 1,2,\ j = 2,3,\cdots,6)$, and $\eta(>0)\in R$, we determine it later.

When the time derivative of V_1 and V_2 along the trajectory of system (2.1) are calculate, we obtain

$$
\begin{aligned}
V_1'(t) &= e^{2kt}(2k\rho_1 D_1^2 + 2\rho_1 D_1 D_1')\\
&= 2e^{2kt}k\rho_1 D_1^2 + 2e^{2kt}\left[D_1, \sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} x(s)ds, 0\right]\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\rho_1&\rho_2&\rho_3\\0&\rho_4&\rho_5\\0&0&\rho_6\end{bmatrix}\begin{bmatrix}D_1'\\0\\0\end{bmatrix}.
\end{aligned}
$$

Benefited from the formula, $2x-\sum_{i=1}^{2}x(t-\tau_i(t)) = \sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds$, we have

$$
\begin{aligned}
V_1'(t) =& 2e^{2kt}k\rho_1 D_1^2\\
&+2e^{2kt}\left[D_1, \sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} x(s)ds, -\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} x'(s)ds + 2x - \sum_{i=1}^{2}x(t-\tau_i(t))\right]\begin{bmatrix}\rho_1&\rho_2&\rho_3\\0&q_4&q_5\\0&0&\rho_6\end{bmatrix}\\
&\times\begin{bmatrix} -a(t)h_1(x)x+\sum_{i=1}^{2}b_i(t)\tanh x(t-\tau_i(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}c(t,s)g_1(x(s))x(s)ds\\ -D_1+x+\sum_{i=1}^{2}p_i(t)x(t-\tau_i(t))\\ \sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds-2x+\sum_{i=1}^{2}x(t-\tau_i(t))\end{bmatrix}\\
=&2e^{2kt}k\rho_1 D_1^2+2e^{2kt}\Bigg[D_1\rho_1, D_1\rho_2+\rho_4\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds,\\
&D_1\rho_3+\rho_5\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds-\rho_6\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds+2\rho_6 x-\rho_6\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(t-\tau_i(t))\Bigg]\\
&\times\begin{bmatrix} -a(t)h_1(x)x+\sum_{i=1}^{2}b_i(t)\tanh x(t-\tau_i(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}c(t,s)g_1(x(s))x(s)ds\\ -D_1+x+\sum_{i=1}^{2}p_i(t)x(t-\tau_i(t))\\ \sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds-2x+\sum_{i=1}^{2}x(t-\tau_i(t))\end{bmatrix}\\
=&2e^{2kt}k\rho_1 D_1^2+2e^{2kt}\Bigg\{-D_1\rho_1 a(t)h_1(x)x+D_1\rho_1\sum_{i=1}^{2}b_i(t)\tanh x(t-\tau_i(t))\\
&+D_1\rho_1\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}c(t,s)g_1(x(s))x(s)ds-D_1^2\rho_2+D_1\rho_2 x\\
&+D_1\rho_2\sum_{i=1}^{2}p_i(t)x(t-\tau_i(t))-D_1\rho_4\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds+\rho_4 x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\rho_4\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\sum_{i=1}^{2}p_i(t)x(t-\tau_i(t))+D_1\rho_3\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds-2D_1\rho_3x\\
&+D_1\rho_3\sum_{i=1}^{2}x(t-\tau_i(t))+\rho_5\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds-2\rho_5x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\\
&+\rho_5\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\sum_{i=1}^{2}x(t-\tau_i(t))-\rho_6\left[\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\right]^2+2\rho_6x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\\
&-\rho_6\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\sum_{i=1}^{2}x(t-\tau_i(t))+2\rho_6x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds-4\rho_6x^2\\
&+2\rho_6x\sum_{i=1}^{2}x(t-\tau_i(t))-\rho_6\sum_{i=1}^{2}x(t-\tau_i(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\\
&+2\rho_6x\sum_{i=1}^{2}x(t-\tau_i(t))-\rho_6\left[\sum_{i=1}^{2}x(t-\tau_i(t))\right]^2\Bigg\}\\
=&e^{2kt}\Big\{(2k\rho_1-2\rho_2)D_1^2+2[-\rho_1a(t)h_1(x)+\rho_2-2\rho_3]xD_1-8\rho_6x^2\\
&+2D_1\rho_1\sum_{i=1}^{2}b_i(t)\tanh x(t-\tau_i(t))+2D_1\rho_1\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}c(t,s)g_1(x(s))x(s)ds\\
&+2D_1\rho_2\sum_{i=1}^{2}p_i(t)x(t-\tau_i(t))-2D_1\rho_4\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds+2\rho_4x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\\
&+2\rho_4\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\sum_{i=1}^{2}p_i(t)x(t-\tau_i(t))+2D_1\rho_3\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\\
&+2D_1\rho_3\sum_{i=1}^{2}x(t-\tau_i(t))+2\rho_5\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\\
&-4\rho_5x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds+2\rho_5\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\sum_{i=1}^{2}x(t-\tau_i(t))\\
&-2\rho_6\left[\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\right]^2+8\rho_6x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\\
&-4\rho_6\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\sum_{i=1}^{2}x(t-\tau_i(t))+8\rho_6x\sum_{i=1}^{2}x(t-\tau_i(t))\\
&-2\rho_6\left[\sum_{i=1}^{2}x(t-\tau_i(t))\right]^2\Bigg\}.
\end{aligned}
$$

Taking into account conditions (2.2), (2.3) and Lemma 2.2 by $\left|\sum_{i=1}^{2}p_i(t)\right|\leqslant p_0<1$, we have

$$
\begin{aligned}
V_1'(t)\leqslant e^{2kt}\Big\{&(2k\rho_1-2\rho_2)D_1^2+2[-\rho_1a_1n_1+\rho_2-2\rho_3]xD_1\\
&+2\rho_1b_{21}D_1\tan hx(t-\tau_1(t))+2\rho_1b_{22}D_1\tan hx(t-\tau_2(t))
\end{aligned}
$$

$$
\begin{aligned}
&+2(\sigma\rho_1 m_2-\rho_4)D_1\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds+2(\rho_2 p_{21}+\rho_3)D_1 x(t-\tau_1(t))\\
&+2(\rho_2 p_{22}+\rho_3)D_1 x(t-\tau_2(t))+2\rho_4 x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\\
&+2(\rho_4 p_{21}+\rho_5)x(t-\tau_1(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds+2\rho_5\left[\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\right]^2\\
&+2(\rho_4 p_{22}+\rho_5)x(t-\tau_2(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds+2\rho_3 D_1\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\\
&+2\rho_5\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds-2\rho_6\left[\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\right]^2\\
&+8\rho_6 x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds-4\rho_6 x(t-\tau_1(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\\
&-4\rho_6 x(t-\tau_2(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds+8\rho_6 xx(t-\tau_1(t))+8\rho_6 xx(t-\tau_2(t))\\
&-2\rho_6 x^2(t-\tau_1(t))-4\rho_6 x(t-\tau_1(t))x(t-\tau_2(t))-2\rho_6 x^2(t-\tau_2(t))\Big\}.
\end{aligned}\tag{2.8}
$$

$$
\begin{aligned}
V_2'(t)=&\alpha e^{2k(t+\tau_1)}x^2-\alpha e^{2k(t+\tau_1-\tau_1(t))}(1-\tau_1'(t))x^2(t-\tau_1(t))\\
&+\alpha e^{2k(t+\tau_2)}x^2-\alpha e^{2k(t+\tau_2-\tau_2(t))}(1-\tau_2'(t))x^2(t-\tau_2(t))\\
&+b_1'(t)\int_{t-\tau_1(t)}^{t}e^{2k(s+\tau_1)}\tanh^2 x(s)ds+b_1(t)e^{2k(t+\tau_1)}\tanh^2 x\\
&-b_1(t)e^{2k(t+\tau_1-\tau(t))}(1-\tau_1'(t))\tanh^2 x(t-\tau_1(t))\\
&+b_2'(t)\int_{t-\tau_1(t)}^{t}e^{2k(s+\tau_2)}\tanh^2 x(s)ds+b_2(t)e^{2k(t+\tau_2)}\tanh^2 x\\
&-b_2(t)e^{2k(t+\tau_2-\tau(t))}(1-\tau_2'(t))\tanh^2 x(t-\tau_2(t)).
\end{aligned}
$$

Using the conditions (1.5) and (2.2) with applying the estimate $\tanh^2 x\leqslant x^2$, we have

$$
\begin{aligned}
V_2'(t)\leqslant e^{2kt}\Big\{&\Big[(\alpha+b_{21})e^{2k\tau_1}+(\alpha+b_{22})e^{2k\tau_2}\Big]x^2\\
&-\alpha(1-\delta_1)x^2(t-\tau_1(t))-\alpha(1-\delta_2)x^2(t-\tau_2(t))\\
&-b_{11}(1-\delta_1)\tanh x^2(t-\tau_1(t))-b_{12}(1-\delta_2)\tanh x^2(t-\tau_2(t))\Big\}.
\end{aligned}\tag{2.9}
$$

Combining equations (2.8) and (2.9), we can readily get

$$
\begin{aligned}
V_1'(t)+V_2'(t)\leqslant e^{2kt}\Big\{&(2k\rho_1-2\rho_2)D_1^2+2[-\rho_1 a_1 n_1+\rho_2-2\rho_3]D_1 x\\
&+2(\rho_2 p_{21}+\rho_3)D_1 x(t-\tau_1(t))+2(\rho_2 p_{22}+\rho_3)D_1 x(t-\tau_2(t))\\
&+2\rho_1 b_{21}D_1\tan hx(t-\tau_1(t))+2\rho_1 b_{22}D_1\tan hx(t-\tau_2(t))\\
&+2(\sigma\rho_1 m_2-\rho_4)D_1\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds+2\rho_3 D_1\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[2(-4\rho_6+\rho_5)+(\alpha+b_{21})e^{2k\tau_1}+(\alpha+b_{22})e^{2k\tau_2}\Big]x^2\\
&+8\rho_6 x(x(t-\tau_1(t))+x(t-\tau_2(t)))+2\rho_4 x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\\
&+8\rho_6 x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds-[\alpha(1-\delta_1)+2\rho_6]x^2(t-\tau_1(t))\\
&-4\rho_6 x(t-\tau_1(t))x(t-\tau_2(t))+2(\rho_4 p_{21}+\rho_5)x(t-\tau_1(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\\
&-4\rho_6 x(t-\tau_1(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds-[\alpha(1-\delta_2)+2\rho_6]x^2(t-\tau_2(t))\\
&+2(\rho_4 p_{22}+\rho_5)x(t-\tau_2(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds-4\rho_6 x(t-\tau_2(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\\
&-b_{11}(1-\delta_1)\tanh x^2(t-\tau_1(t)-b_{12}(1-\delta_2)\tanh x^2(t-\tau_2(t)\Big\}\\
&+2\rho_5\left[\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\right]^2+2\rho_5\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\\
&-2\rho_6\left[\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\right]^2\\
=&e^{2kt}\pi^t\Xi\pi(t),
\end{aligned}
$$

where $\pi(t)=\Big[D_1,\ x,\ x(t-\tau_1(t)),\ x(t-\tau_2(t)),\ \tanh x(t-\tau_1(t)),\ \tanh x(t-\tau_2(t)),\ \sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds,$ $\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\Big]^T$ and Ξ is defined in Eq. (2.7). The requirement $\Xi<0$ when taken into consideration, it is clear that

$$V_1'(t)+V_2'(t)\leqslant e^{2kt}\pi^T(t)\Xi\pi(t)<0.$$

Thus, there is a positive constant λ such that

$$
\begin{aligned}
V_1'(t)+V_2'(t)\leqslant -\lambda e^{2kt}\Big(&\|D_1\|^2+\|x\|^2+\|x(t-\tau_1(t)\|^2+\|x(t-\tau_2(t)\|^2\\
&+\|\tanh x(t-\tau_1(t))\|^2+\|\tanh x(t-\tau_2(t))\|^2+\left\|\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\right\|^2\\
&+\|\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x'(s)ds\|^2\Big)\\
\leqslant& -\lambda e^{2kt}\|x\|^2.
\end{aligned}
$$

Calculating the derivative of V_3 along the trajectory of system (2.1), we get

$$
\begin{aligned}
V_3'(t)=&2e^{2kt}\eta(D_1D_1'+kD_1^2)\\
=&2e^{2kt}\eta\Big\{\Big[x+\sum_{i=1}^{2}p_i(t)x(t-\tau_i(t))\Big]\times\Big[-a(t)h_1(x)x
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{2} b_i(t)\tanh x(t-\tau_i(t)) + \sum_{i=1}^{2}\int_{y-\tau_i(t)}^{t} c(t,s)g_1(x(s))x(s)ds\Bigg]\\
&+k\Bigg[x+\sum_{i=1}^{2} p_i(t)x(t-\tau_i(t))\Bigg]^2\Bigg\}\\
=&2e^{2kt}\eta\Bigg\{-a(t)h_1(x)x^2 + b_1(t)x\tanh x(t-\tau_1(t))\\
&+b_2(t)x\tanh x(t-\tau_2(t)) + x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} c(t,s)g_1(x(s))x(s)ds\\
&-a(t)h_1(x)p_1(t)xx(t-\tau_1(t)) - a(t)h_1(x)p_2(t)xx(t-\tau_2(t))\\
&+b_1(t)p_1(t)x(t-\tau_1(t))\tanh x(t-\tau_1(t))\\
&+b_1(t)p_2(t)x(t-\tau_2(t))\tanh x(t-\tau_1(t))\\
&+b_2(t)p_1(t)x(t-\tau_1(t))\tanh x(t-\tau_2(t))\\
&+b_2(t)p_2(t)x(t-\tau_2(t))\tanh x(t-\tau_2(t))\\
&+p_1(t)x(t-\tau_1(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} c(t,s)g_1(x(s))x(s)ds\\
&+p_2(t)x(t-\tau_2(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} c(t,s)g_1(x(s))x(s)ds\\
&+kx^2 + kp_1^2(t)x^2(t-\tau_1(t)) + kp_2^2(t)x^2(t-\tau_2(t))\\
&+2kp_1(t)xx(t-\tau_1(t)) + 2kp_2(t)xx(t-\tau_2(t))\\
&+2kp_1(t)p_2(t)x(t-\tau_1(t))x(t-\tau_2(t))\Bigg\}.
\end{aligned}
$$

Utilizing conditions (2.2), (2.3) and by $\left|\sum_{i=1}^{2} p_i(t)\right| \leqslant p_0 < 1$, we obtain

$$
\begin{aligned}
V_3'(t) \leqslant& e^{2kt}\eta\Bigg\{-2a_1n_1x^2 + 2b_{21}(t)x\tanh x(t-\tau_1(t)) + 2b_{22}(t)x\tanh x(t-\tau_2(t))\\
&+2\sigma m_2 x\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} x(s)ds - 2a_1n_1p_{11}xx(t-\tau_1(t)) - 2a_1n_1p_{12}xx(t-\tau_2(t))\\
&+2b_{21}p_{21}(t)x(t-\tau_1(t))\tanh x(t-\tau_1(t))\\
&+2b_{21}p_{22}(t)x(t-\tau_2(t))\tanh x(t-\tau_1(t))\\
&+2b_{22}p_{21}(t)x(t-\tau_1(t))\tanh x(t-\tau_2(t))\\
&+2b_{22}p_{22}(t)x(t-\tau_2(t))\tanh x(t-\tau_2(t))\\
&+2p_{21}\sigma m_2 x(t-\tau_1(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} x(s)ds\\
&+2p_{22}\sigma m_2 x(t-\tau_2(t))\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} x(s)ds\\
&+2kx^2 + 2kp_{21}^2x^2(t-\tau_1(t)) + 2kp_{22}^2x^2(t-\tau_2(t))\\
&+4kp_{21}xx(t-\tau_1(t)) + 4kp_{22}(t)xx(t-\tau_2(t))\\
&+4kp_{21}p_{22}(t)x(t-\tau_1(t))x(t-\tau_2(t))\Bigg\}.
\end{aligned}
$$

From the Lemma 2.2, we have

$$\begin{aligned} V_3'(t) \leqslant & e^{2kt}\eta\Bigg\{\Bigg[2k-2a_1n_1+|b_{21}|^2+|b_{22}|^2+|m_2\sigma|^2+|a_1n_1p_{11}|^2+|a_1n_1p_{12}|^2 \\ & +4|kp_{21}|^2+4|kp_{22}|^2\Bigg]x^2+\Bigg[2kp_{21}^2+|b_{21}p_{21}|^2+|b_{22}p_{21}|^2+|p_{21}m_2\sigma|^2 \\ & +4|kp_{21}p_{22}|^2+2\Bigg]x^2(t-\tau_1(t))+\Bigg[2kp_{22}^2+|b_{21}p_{22}|^2+|b_{22}p_{22}|^2 \\ & +|p_{22}m_2\sigma|^2+3\Bigg]x^2(t-\tau_2(t))+3\tanh^2 x(t-\tau_1(t)) \\ & +3\tanh^2 x(t-\tau_2(t))+3\left[\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x(s)ds\right]^2\Bigg\}. \end{aligned}$$

Let us choose the constant η as

$$\eta=\begin{cases} \frac{\lambda}{2}\min\left\{\frac{1}{\xi_1},\frac{1}{\xi_2},\frac{1}{3}\right\}, & \text{if } \psi\leqslant 0, \\ \frac{\lambda}{2}\min\left\{\frac{1}{\xi_1},\frac{1}{\xi_2},\frac{1}{\psi},\frac{1}{3}\right\}, & \text{if } \psi>0, \end{cases}$$

where
$\xi_1=2kp_{21}^2+|b_{21}p_{21}|^2+|b_{22}p_{21}|^2+|p_{21}m_2\sigma|^2+4|kp_{21}p_{22}|^2+2$, $\xi_2=|p_{22}m_2\sigma|^2+|b_{22}p_{22}|^2+2kp_{22}^2+3$ and $\psi=2k-2a_1n_1+|b_{21}|^2+|b_{22}|^2+|m_2\sigma|^2+|a_1n_1p_{11}|^2+|a_1n_1p_{12}|^2+4|kp_{21}|^2+4|kp_{22}|^2$.
Hence, we can readily achieve the following inequality

$$V_1'(t)+V_2'(t)+V_3'(t)\leqslant -\frac{\lambda}{2}e^{2kt}\|x\|^2<0.$$

Since $V'(t)$ is negative-definite and $0\leqslant\tau_i(t)\leqslant\tau_i$, $(i=1,2)$, then, $V(x)\leqslant V(x(0))$ for all $t\geqslant 0$, with

$$\begin{aligned} V(x(0)) = & V_1(x(0))+V_2(x(0))+V_3(x(0)) \\ = & \rho_1\left[x(0)+\sum_{i=1}^{2}p_i(0)x(-\tau_i(0))\right]^2 \\ & +\alpha\sum_{i=1}^{2}\int_{-\tau_i(0)}^{0}e^{2k(s+\tau_i)}x^2(s)ds+\sum_{i=1}^{2}b_i(0)\int_{-\tau_i(0)}^{0}e^{2k(s+\tau_i)}\tanh^2 x(s)ds \\ & +\eta\left[x(0)+\sum_{i=1}^{2}p_i(0)x(-\tau_i(0))\right]^2 \\ \leqslant & \rho_1\left[1+\sum_{i=1}^{2}p_i(0)\right]^2\|x_0\|_s^2+\alpha\sum_{i=1}^{2}\int_{-\tau_i(0)}^{0}e^{2k(s+\tau_i)}\left(\sup_{-\tau_i\leqslant s\leqslant 0}\|x(s)\|\right)^2 ds \\ & +b_{21}\int_{-\tau_1(0)}^{0}e^{2k(s+\tau_1)}\left(\sup_{-\tau_1\leqslant s\leqslant 0}\|x(s)\|\right)^2 ds+b_{22}\int_{-\tau_2(0)}^{0}e^{2k(s+\tau_2)}\left(\sup_{-\tau_2\leqslant s\leqslant 0}\|x(s)\|\right)^2 ds \\ & +\eta\left[1+\sum_{i=1}^{2}p_i(0)\right]^2\|x_0\|_s^2 \\ \leqslant & \rho_1\left[1+\sum_{i=1}^{2}p_i(0)\right]^2\|x_0\|_s^2+\alpha e^{2k\tau_1}\tau_1\|x_0\|_s^2+\alpha e^{2k\tau_2}\tau_2\|x_0\|_s^2 \end{aligned}$$

$$+ b_{21}e^{2k\tau_1}\tau_1\|x_0\|_s^2 + b_{22}e^{2k\tau_2}\tau_2\|x_0\|_s^2 + \eta\left[1+\sum_{i=1}^{2} p_i(0)\right]^2\|x_0\|_s^2$$
$$=\Delta\|x_0\|_s^2,$$

where $\Delta = \rho_1\left[1+\sum_{i=1}^{2} p_i(0)\right]^2 + (\alpha b_{21}e^{2k\tau_1}\tau_1) + (\alpha b_{22}e^{2k\tau_2}\tau_2) + \eta\left[1+\sum_{i=1}^{2} p_i(0)\right]^2$.
From $\eta e^{2kt}\|D_1\|^2 \leqslant V(x) \leqslant \Delta\|x_0\|_s^2$, we obtain $\|D_1\| \leqslant Me^{-kt}$,
where $M = \sqrt{\frac{\Delta}{\eta}}\|x_0\|_s$. Since $D_1 = x + \sum_{i=1}^{2} p_i(t)x(t-\tau_i(t))$, we have

$$\|x\| = \left\|D_1 - \sum_{i=1}^{2} p_i(t)x(t-\tau_i(t))\right\| \leqslant \|D_1\| + \left\|\sum_{i=1}^{2} p_i(t)x(t-\tau_i(t))\right\| \leqslant Me^{-kt} + p_0\left\|\sum_{i=1}^{2} x(t-\tau_i(t))\right\|.$$

Because $\left|\sum_{i=1}^{2} p_i(t)\right| \leqslant p_0 < 1$ and $0 \leqslant \tau_i(t) \leqslant \tau_i$, we can choose sufficiently small positive constant $\vartheta = k < \frac{-\ln p_0}{\tau_i}$ so that $p_0e^{\vartheta\tau_i}$, $i = (1,2) < 1$. Taking into account Proposition 2.3, we have

$$\|x\| \leqslant \left(\|x_0\|_s + \frac{M_i}{1-p_0e^{\vartheta\tau_i}}\right)e^{-\vartheta t}, t \geqslant 0.$$

for $(i = 1,2)$ choosing $\gamma = \max\left\{\|x_0\|_s, \frac{M_i}{1-p_0e^{\vartheta\tau_i}}\right\}$, we have

$$\|x\| \leqslant 2\gamma e^{-\vartheta t},$$

for $(i = 1,2)$. This implies that the zero solution of NVIDE (1.8) is exponential stability. By radially unboundedness, it is also GES with rate of convergence $k = \vartheta > 0$. □

Remark 2.5. It can be easily seen that the zero solution of NVIDE (1.8) is uniformly asymptotically stable, if $k = 0$ and the following criterion holds

$$\Xi = \begin{bmatrix} -2\rho_2 & \Xi_{12} & \Xi_{13} & \Xi_{14} & \rho_1 b_{21} & \rho_1 b_{22} & \Xi_{17} & \rho_3 \\ * & \Xi_{22} & 4\rho_6 & 4\rho_6 & 0 & 0 & \rho_4 & 4\rho_6 \\ * & * & \Xi_{33} & -2\rho_6 & 0 & 0 & \Xi_{37} & -2\rho_6 \\ * & * & * & \Xi_{44} & 0 & 0 & \Xi_{47} & -2\rho_6 \\ * & * & * & * & -b_{11}(1-\delta_1) & 0 & 0 & 0 \\ * & * & * & * & * & -b_{12}(1-\delta_2) & 0 & 0 \\ * & * & * & * & * & * & 2\rho_5 & \rho_5 \\ * & * & * & * & * & * & * & -2\rho_6 \end{bmatrix} < 0,$$

where $\Xi_{12} = -\rho_1 a_1 n_1 + \rho_2 - 2\rho_3$, $\Xi_{13} = \rho_2 p_{21} + \rho_3$, $\Xi_{14} = \rho_2 p_{22} + \rho_3$, $\Xi_{17} = \sigma\rho_1 m_2 - \rho_4$, $\Xi_{22} = 2(-4\rho_6 + \rho_5) + 2\alpha + b_{21} + b_{22}$, $\Xi_{33} = -\alpha(1-\delta_1) - 2\rho_6$, $\Xi_{37} = \rho_4 p_{21} + \rho_5$, $\Xi_{44} = -\alpha(1-\delta_2) - 2\rho_6$, $\Xi_{47} = \rho_4 p_{22} + \rho_5$.

Example 2.6. *For $i = 1$, as a special case NVIDE (1.4), we consider the following nonlinear NVIDE time-varying lag,*

$$\frac{d}{dt}\left[x + \frac{1}{8+t^2}x(t-\tau_1(t))\right] = -(2+\exp(-t))\left[x + \frac{x}{1+x^2}\right] + \frac{1}{3}\tanh x(t-\tau_1(t))$$
$$+ \int_{t-\tau_1(t)}^{t} \exp(-t+s)(x(s)\frac{x(s)}{1+x^4(s)}ds), \quad t \geqslant 0. \tag{2.10}$$

Here,

$$
\begin{aligned}
&D_1(t) = x + \frac{1}{8+t^2}x(t-\tau_1(t)), \;\; p_1(t) = \frac{1}{8+t^2}, \;\; a(t) = 2+\exp(-t),\\
&b_1(t) = \frac{1}{3}, \;\; \tau_1(t) = \frac{\sin^2 t}{5} \leqslant \frac{1}{5}, \;\; \tau_1'(t) = \frac{\sin 2t}{5} \leqslant \frac{1}{5} \leqslant \frac{1}{5} = \delta_1 < 1,\\
&h(x) = x + \frac{x}{1+x^2}, h_1(x) = \begin{cases} 1+\frac{1}{1+x^2}, x \neq 0 \\ \\ h'(0), \quad x = 0 \end{cases}\\
&g(x) = x + \frac{x}{1+x^4}, g_1(x) = \begin{cases} 1+\frac{1}{1+x^4}, x \neq 0 \\ \\ g'(0), \quad x = 0 \end{cases}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&h(0) = 0, n_1 = 1 \leqslant h_1(x) \leqslant 2 = n_2, \;\; g(0) = 0, m_1 = 1 \leqslant g_1(x) \leqslant 2 = m_2\\
&a_1 = 2 \leqslant a(t) = 2+\exp(-t) \leqslant 3 = a_2, \;\; b_{11} = b_1(t) = \frac{1}{3} = b_{21}, \;\; \alpha = \frac{5}{8}, \;\; k = \frac{1}{4},\\
&c(t,s) = \exp(-t+s) \leqslant 1 = \sigma, \;\; p_{11} = 0 \leqslant p_1(t) = \frac{1}{8+t^2} \leqslant \frac{1}{8} = p_{21} = p_0 < 1,\\
&c_1 = c_2 = \frac{1}{3}, \delta_1 = \frac{1}{5}, \delta_2 = \frac{1}{10}, \alpha = \frac{5}{8}, k = \frac{1}{4}\\
&\rho_1 = 1.5 > 0, \;\; \rho_2 = 1.5, \;\; \rho_3 = 0.5, \;\; \rho_4 = 3 \;\; \rho_5 = -3, \rho_6 = 1.5,
\end{aligned}
$$

and

$$
\tilde{\Xi} = \begin{bmatrix}
2k\rho_1 - 2\rho_2 & -\rho_1 a_1 n_1 + \rho_2 - 2\rho_3 & \rho_2 p_{21} + \rho_3 & \rho_1 b_{21} & \sigma\rho_1 m_2 - \rho_4 & \rho_3\\
* & -8\rho_6 + 2\rho_5 + (\alpha + b_{21})e^{2k\tau_1} & 4\rho_6 & 0 & \rho_4 & 4\rho_6\\
* & * & -\alpha(1-\delta_1) - 2\rho_6 & 0 & \rho_4 p_{21} + \rho_5 & -2\rho_6\\
* & * & * & -b_{11}(1-\delta_1) & 0 & 0\\
* & * & * & * & 2\rho_5 & \rho_5\\
* & * & * & * & * & -2\rho_6
\end{bmatrix} < 0, \quad (2.11)
$$

where $\tilde{\Xi}_{11} = -2.25, \tilde{\Xi}_{12} = -2.5, \tilde{\Xi}_{13} = -0.6875, \tilde{\Xi}_{14} = \tilde{\Xi}_{16} = 0.5, \tilde{\Xi}_{15} = 0, \tilde{\Xi}_{22} = -16.9409$, $\tilde{\Xi}_{24} = 0, \tilde{\Xi}_{25} = 3, \tilde{\Xi}_{23} = \tilde{\Xi}_{26} = 6, \tilde{\Xi}_{33} = -3.5, \tilde{\Xi}_{34} = 0, \tilde{\Xi}_{35} = -2.625, \tilde{\Xi}_{36} = -3, \tilde{\Xi}_{44} = -0.26,$, $\tilde{\Xi}_{45} = \tilde{\Xi}_{46} = 0, \tilde{\Xi}_{55} = -6, \tilde{\Xi}_{56} = \tilde{\Xi}_{66} = -3$. *The eigenvalues of matrix* $\tilde{\Xi}$ *defined by* (2.11) *are* $-23.1626, -5.8881, -1.9409, -0.7979, -0.1417$ *and* -0.0197. *Thus, it is clear that all the assumptions of Theorem 2.4 hold. This discussion implies that NVIDE (2.10) is GES.*

When the theoretical solution of the above example (Example 2.6) is examined, it is seen that the zero solution of considered equation in the example is stable after a certain time interval under different initial conditions. This is confirmed by the related simulation result (Figure 1).

3. Conclusion

In this study, we have derived some a new sufficient condition that guaranteeing the GES of the zero solution of a NVIDE with variable lags. Stability criterion has been obtained by constructing an appropriate Lyapunov functional and formulating in matrix inequality form. An example was given to show the effectiveness of the considered equation by MATLAB Simulink. It is observed that the Example 2.6 and simulation result (Fig.1) verify the efficiency and accuracy of the theoretical result of this study. The obtained result extends and generalizes the existing ones in the literature.

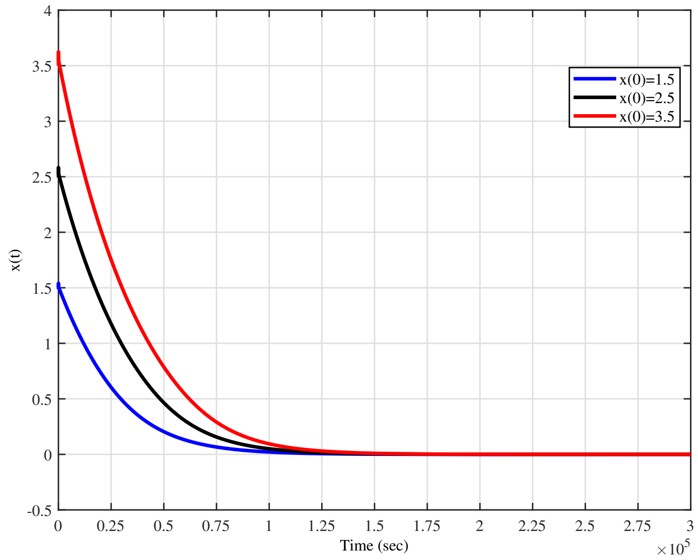

Figure 1: Trajectories of $x(t)$ of Eq. 2.10 in Example, when $\tau_1(t) = \frac{\sin^2 t}{5}, \ \ t \geqslant 0$.

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