Available online at www.isr-publications.com/mns
Math. Nat. Sci., 6 (2020), 15–19
Research Article

Online: ISSN 2600-7665
Mathematics in Natural Science
An International Journal
Journal Homepage: www.isr-publications.com/mns

Lacunary statistical boundedness of measurable functions

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Abstract

In this article, we present new concept of lacunary statistical boundedness by taking nonnegative measurable real valued function on $(1, \infty)$. Additionally, we examine some inclusion theorems.

Keywords: Lacunary sequence, statistical boundedness, measurable functions, lacunary statistical bounded.

2010 MSC: 40D25, 40G15.

1. Introduction

Fast [5] put forward the idea of statistical convergence in 1951. After the paper of Fridy in [7] statistical convergence has been discussed in the several branches of mathematics such as the theory of number in [4], measure theory in [9], probability theory in [3]. Additionally, further investigations that are linked with summability theory were presented by Connor [2], Rath and Tripathy [11], Fridy and Orhan [8] and many others. Statistical convergence depends on the density of subset of the set $\mathbb{N} = \{1, 2, 3, \cdots\}$. Let us consider the natural density of a subset S of $\mathbb{N}$ by

$$\delta(S) = \lim_{t\to\infty} \frac{1}{t} |\{r \leqslant t : r \in S\}|,$$

where the vertical bars show the total number of elements in the enclosed set. We shall consider subsets of $\mathbb{N}$ that have natural density zero [5]. To enable this, Fridy in [7] introduced the following definition.

Definition 1.1. A sequence $y = (y_r)$ (of real or complex numbers) is said to be statistically convergent to some number ξ if for every $\varepsilon > 0$,

$$\delta(\{r \in \mathbb{N} : |y_r - \xi| \geqslant \varepsilon\}).$$

Whenever this occurs, we can write $st - \lim_r y_r = \xi$.

Following Fridy's work, in [7] the definition of lacunary density was presented as follows:

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doi: 10.22436/mns.06.01.03
Received: 2020-05-14 Revised: 2020-06-16 Accepted: 2020-06-23

Definition 1.2. By a lacunary sequence $\theta = \{r_s\}$, $s = 0, 1, 2, 3, \cdots$ where $r_0 = 0$, we shall an increasing sequence of non-negative integers with $r_s - r_{s-1} \to \infty$ as $s \to \infty$. The intervals determined by θ will be denoted by $J_s = (r_{s-1}, r_s]$ and let $\gamma_s = \gamma_s - \gamma_{s-1}$ and $\zeta_r = \frac{r_s}{r_{s-1}}$. For $\theta = \{r_s\}$, we define the lacunary density by

$$\delta^{\theta}(S) = \lim_{s\to\infty} \frac{1}{r_s} |\{r \in J_s : r \in S\}|.$$

While the work in sequence continued, the notion of strongly summable function was introduced by Borwein in [1], and also Nuray and Aydin [10] introduced the concept of lacunary statistical convergence of measurable functions. The primary aim of this article is to introduce the concept of lacunary statistical boundedness functions by taking nonnegative measurable real valued function on $(1, \infty)$. Additionally, we examine some inclusion theorems.

2. Main Results

In this part, we shall present new definitions. Throughout this paper $h(\psi)$ shall be real valued measurable function in the interval $(1, \infty)$.

Definition 2.1. A function $h(\xi)$ is said to be statistical bounded if there exists some constant Λ such that

$$\delta(\{\psi : |h(\psi)| > \Lambda\}) = 0, \text{ in other word, } |h(\xi)| \leqslant \Lambda \text{ almost all } \psi .$$

We will denote the set of all statistically bounded functions by $S_f(B)$.

Bounded functions are clearly statistically bounded as the empty set has zero natural density, but the converse is not correct, as the following instance indicates.

Example 2.2. *Let us consider the function*

$$h(\psi) = \begin{cases} \psi, & \text{if } \psi \text{ is a square;} \\ 0, & \text{if } \psi \text{ is not a square.} \end{cases}$$

$h(\psi)$ *is statistically bounded. It is clear that*

$$\delta\left(\left\{\psi : \left| h(\psi) > \frac{1}{3} \right|\right\}\right) = 0.$$

However, $h(\psi)$ *is not a bounded function.*

Proposition 2.3. *Every convergent functions is statistically bounded.*

Even though a statistically convergent function does not need to be bounded, the following proposition demonstrates that every statistical convergent function is statistically bounded.

Proposition 2.4. *Every statistically convergent function is statistically bounded.*

Proof. Let $h(\psi)$ statistically converges to ξ, and for $\varepsilon > 0$,

$$\delta(\{\psi : |h(\psi) - \xi| > \varepsilon\}) = 0,$$

because of the containment $\{\psi : |h(\psi)| > |\xi| + \varepsilon\} \subset \{\psi : |h(\psi) - \xi| > \varepsilon\}$, we can say $|h(\psi)| \leqslant |\xi| + \varepsilon$ (almost all ψ). □

The following example demonstrates that every function is not statistically bounded.

Example 2.5. *Let us consider a real valued function*

$$h(\psi) = \begin{cases} \psi, & \text{if } \psi = 1, 2, \cdots; \\ 0, & \text{otherwise.} \end{cases}$$

Then for any $\psi \geqslant 0$, *we have* $\{\psi : |h(\psi)| > \xi\} = \mathbf{N} - A$, *where* A *is a finite subset of* $\mathbf{N}$, *so* $\delta(\{\psi : |h(\psi)| > L\}) = 1$. *Hence,* $h(\psi)$ *is not statistically bounded.*

Definition 2.6. Let $\theta = \{r_s\}$ be a lacunary sequence and a real valued function $h(\psi)$ is said to be lacunary statistical bounded or S_θ^f-bounded if there exists $\Lambda > 0$ such that

$$\lim_{s\to\infty} \frac{1}{r_s} |\{\psi \in J_s : |h(\psi)| > \Lambda\}| = 0,$$

i.e.,

$$\delta^\theta(\{\psi \in \mathbb{N} : |h(\psi)| > \Lambda\}) = 0, \text{ i.e., } |h(\psi)| \leqslant \Lambda \ (\text{almost all } \psi \text{ with respect to } \theta).$$

In this case, we denote the set of all S_θ^f-bounded functions.

Theorem 2.7. *Every lacunary statistical convergent function is lacunary statistical bounded, but not conversely.*

Proof. Let $h(\psi) \to \xi \left(S_\theta^f\right)$. Then for each $\varepsilon > 0$, we obtain the following

$$\lim_{s\to\infty} \frac{1}{r_s} |\{\psi \in J_s : |h(\psi) - \xi| \geqslant \varepsilon\}| = 0.$$

The result now follows from the fact that

$$\{\psi \in J_s : |h(\psi)| \geqslant |\xi| + \varepsilon\} \subset \{\psi \in J_s : |h(\psi) - \xi| \geqslant \varepsilon\}.$$

It is easy be confirmed that lacunary statistical bounded function $\left\{(-1)^\psi\right\}$ is not lacunary statistical convergent. □

Theorem 2.8. *For a given lacunary sequence* $\theta = \{r_s\}$, *a real valued function* $h(\psi)$ *is lacunary statistical bounded if and only if there exists a bounded function* $g(\psi)$ *such that* $h(\psi) = g(\psi)$ *almost all* ψ *with respect to* θ.

Proof. Assume that $h(\psi)$ is a lacunary statistical bounded function. Then there exists $\Lambda \geqslant 0$ such that $\delta^\theta(\kappa) = 0$ where $\kappa = \{\psi \in \mathbb{N} : |h(\psi)| > \Lambda\}$. Let

$$g(\psi) = \begin{cases} h(\psi), & \text{if } \psi \notin \kappa; \\ 0, & \text{otherwise.} \end{cases}$$

Then $g(\psi) \in S^f(B)$, and $h(\psi) = g(\psi)$ almost all ψ with respect to θ. On the contrary, because of $g(\psi) \in S_f(B)$ there exists $\xi \geqslant 0$ such that $|g(\psi)| \leqslant \xi$ for all $\psi \in \mathbb{N}$. Let $R = \{\psi \in \mathbb{N} : h(\psi) \neq g(\psi)\}$. As $\delta^\theta(A) = 0$, so $|h(\psi)| \leqslant \xi$ almost all ψ with respect to θ. □

Theorem 2.9. *For any lacunary sequence* θ, $S_f(B) \subset S_\theta^f(B)$ *if and only if* $\liminf_s \zeta_s > 1$.

Proof. Provided that $\liminf_s \zeta_s > 1$, then there exists $\tilde{\delta} > 0$ such that $\zeta_s \geqslant 1 + \tilde{\delta}$ for sufficiently large s. Since $\gamma_s = r_s - r_{s-1}$, so we have $\frac{\gamma_s}{r_s} \geqslant \frac{\tilde{\delta}}{1+\tilde{\delta}}$ and $\frac{r_{s-1}}{\gamma_s} \leqslant \frac{1}{\tilde{\delta}}$. For $h(\psi) \in S_\theta^f(B)$, there exists $\Lambda > 0$ such that $\lim_{s\to\infty} \frac{1}{\gamma_s} |\{\psi \in J_s : |h(\psi)| > \Lambda\}| = 0$. Now for sufficiently large s, we obtain the following

$$\frac{1}{r_s} |\{\psi \leqslant r_s : |h(\psi)| > \Lambda\}| \geqslant \frac{\gamma_s}{r_s} \frac{1}{\gamma_s} |\{\psi \in J_s : |h(\psi)| > \Lambda\}|$$

$$\geqslant \frac{\widetilde{\delta}}{1+\widetilde{\delta}} \frac{1}{\gamma_s} \left|\{\psi \in J_s : |h(\psi)| > \Lambda\}\right|.$$

Therefore, sufficient condition is proved.

Suppose that $\liminf_s \zeta_s = 1$. Let us now select a subsequence $\{r_{s_j}\}$ of $\theta = \{r_s\}$ satisfying

$$\frac{r_{s_j}}{r_{s_j-1}} < 1 + \frac{1}{j} \text{ and } \frac{r_{s_j-1}}{r_{s_{j-1}}} > j, \text{ where } s_j \geqslant s_{j-1} + 2.$$

Let us define $h(\psi)$ by

$$h(\psi) = \begin{cases} \psi, \text{ if } \psi \in J_{s_j} & \text{for some } j = 1, 2, 3, \cdots; \\ 0, & \text{otherwise.} \end{cases}$$

Now for any $\Lambda > 0$, there exists $j_0 \in \mathbb{N}$, indeed there are infinitely many j_0, such that $r_{s_{j_0-1}} > \Lambda$ and we have

$$\frac{1}{\gamma_{s_{j_0}}} \left|\left\{\psi \in J_{s_{j_0}} : |h(\psi)| > \Lambda\right\}\right| \geqslant \frac{1}{\gamma_{s_{j_0}}} \left|\left\{\psi \in J_{s_{j_0}} : |h(\psi)| > r_{s_{j_0-1}}\right\}\right| = 1,$$

i.e.,

$$\frac{1}{\gamma_{s_j}} \left|\left\{\psi \in J_{s_j} : |h(\psi)| > \Lambda\right\}\right| = 1,$$

for all $j \geqslant j_0$. Also, for $s \neq s_j$,

$$\frac{1}{\gamma_s} \left|\{\psi \in J_s : |h(\psi)| > \Lambda\}\right| = 0.$$

Hence, $h(\psi) \notin S_\theta^f(B)$.

Now for any τ sufficiently large integer, we can find the unique j for which $r_{s_j-1} < \tau \leqslant r_{s_{j+1}-1}$ and we can write

$$\frac{1}{\tau}\left|\left\{\psi \leqslant \tau : |h(\psi)| > \frac{1}{2}\right\}\right| \leqslant \frac{r_{s_{j-1}} + \gamma_{s_j}}{r_{s_j-1}} < \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.$$

Since $\tau \to \infty$ implies $j \to \infty$, we have $h(\psi) \in S_f(B)$. □

Remark 2.10. A real valued function $h(\psi)$, which was used to present in the necessity part of above theorem, is an instance of a statistical bounded function which is not lacunary statistical bounded.

Theorem 2.11. *For any lacunary sequence* θ, $S_\theta^f(B) \subset S_f(B)$ *if and only if* $\limsup_s \zeta_s < \infty$.

Proof. The sufficiency part of this theorem can be proved following a similar technique to Lemma 3 of [6]. On the other hand, we assume that $\limsup_s \zeta_s = \infty$. Let us now select a subsequence $\{r_{s_j}\}$ of the lacunary sequence θ such that $\zeta_{s_j} > j$. Let us define $h(\psi)$ as

$$h(\psi) = \begin{cases} \psi, & \text{if } r_{s_j-1} < j \leqslant 2r_{s_j-1}; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\eta_{s_j} = \frac{1}{\gamma_{s_j}}\left|\left\{\psi \in J_{s_j} : |h(\psi)| > \frac{1}{2}\right\}\right| = \frac{r_{s_j-1}}{r_{s_j} - r_{s_j-1}} < \frac{1}{j-1},$$

and if $s \neq s_j$, $\eta_s = 0$. Thus, $h(\psi) \in S_\theta^f(B)$. On the other hand, for any real $\Lambda > 0$, there exists some j_0 such that $r_{s_j-1} > \Lambda$ for all $j \geqslant j_0$;

$$\frac{1}{2r_{s_j-1}}\left|\left\{\psi \leqslant 2r_{s_j} : |h(\psi)| > \Lambda\right\}\right| \geqslant \frac{1}{2r_{s_j-1}}\left|\left\{\psi \leqslant 2r_{s_j-1} : |h(\psi)| > r_{s_j-1}\right\}\right| \geqslant \frac{1}{2},$$

and this is true for all $j \geqslant j_0$. Thus, $h(\psi) \notin S_f(B)$. □

Remark 2.12. A real valued function $h(\psi)$, which was used to present in the necessity part of above theorem, is an example of a lacunary statistical bounded function which is not statistical bounded.

Combining Theorem 2.9 and Theorem 2.11, we obtain the following final two theorems.

Theorem 2.13. *Let* θ *be a lacunary sequence. Then* $S_\theta^f(B) = S_f(B)$ *if and only if* $1 < \liminf_s \zeta_s \leqslant \limsup_s \zeta_s < \infty$.

Theorem 2.14. $S_f(B) = \bigcap_{\liminf_s \zeta_s > 1} S_\theta^f(B) = \bigcup_{\limsup_s \zeta_s < \infty} S_\theta^f(B)$.

Proof. Due to Theorem 2.9, we can say that $S_f(B) \subset \cap_{\liminf_s \zeta_s > 1} S_\theta^f(B)$. We assume that if $h(\psi) = \cap_{\liminf_s \zeta_s > 1} S_\theta^f(B)$, but $h(\psi) \notin S_f(B)$, then we have $h(\psi) \in S_\theta^f(B)$ for all $\theta = \{r_s\}$ with $\liminf_s \xi_s > 1$. If we take $\theta = \{2^s\}$, then because of Theorem 2.13 we have $S_\theta^f(B) = S_f(B)$, so $h(\psi) \notin S_f(B)$. In a manner similar to sufficient part, the necessity part can be proved. Therefore, the proof of this theorem is omitted. □

3. Conclusion

Some summability methods were studied by several authors. However by considering measurable functions the more general concept, i.e lacunary statistical bounded function has not been studied so far. Hence, this paper is filled up a gap in the literature.

References

[1] D. Borwein, *Linear functionals connected with strong Cesáro summability*, Journal London Math. Soc., **40** (1965), 628–634. 1

[2] J. S. Connor, *The statistical and strong p−Cesáro convergence of sequences*, Analysis, **8** (1988), 47–63. 1

[3] P. Das, S. Ghosal, S. Som, *Statistical convergence of order α in probability*, Arab. J. Math. Sci., **21** (2015), 253–265. 1

[4] P. Erdös, G. Tenenbaum, *Surles densities de certaines suites d'entiers*, Proc. Lond. Math. Soc., **3 (59)** (1989), 417–438. 1

[5] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244. 1

[6] A. R. Freedman, J. J. Sember, M. Raphael, *Some Cesáro-type summability spaces*, Proc. Lond. Math. Soc., **37** (1978), 508–520. 2

[7] J. A. Fridy, *On statistical convergence*, Analysis, **5** (1985), 301–313. 1, 1

[8] J. A. Fridy, C. Orhan, *Lacunary statistical convergence*, Pacific J. Math., **160** (1993), 43–51. 1

[9] H. I. Miller, *A measure theoretical subsequence characterization of statistical convergence*, Trans. Amer. Math. Soc., **347** (1995), 1811–1819. 1

[10] F. Nuray, B. Aydin, *Strongly summable and statistically convergent functions*, Informacines Technologijos Ir Valdymas, **1 (30)** (2004), 74–76. 1

[11] D. Rath, B. C. Tripathy, *On statistically convergent and statistically Cauchy sequences*, Indian J. Pure Appl. Math., **25 (4)** (1994), 381–386. 1