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Fixed points of generalized $F-H-\phi-\psi-\varphi-$ weakly contractive mappings

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Abstract

We introduce the notion of generalized $F-H-\phi-\psi-\varphi-$ weakly contractive mappings and prove the existence of fixed points of such mappings in complete metric spaces. We draw some corollaries and provide examples in support of our main results. Our results extend the results of Cho [S. Cho, Fixed Point Theory Appl., **2018** (2018), 18 pages] and Choudhury, Konar, Rhoades and Metiya [B. S. Choudhury, P. Konar, B. E. Rhoades, N. Metiya, Nonlinear Anal., **74** (2011), 2116–2126] in the sense that the control function that we used in our results need not have monotonicity property.

Keywords: $\alpha-$admissible, $\mu-$subadmissible, $C-$class function, the pair (F, H) is upclass of type I, the pair (F, H) is special upclass of type I.

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1. Introduction

In 1922, Banach established a theorem which is known as the Banach contraction principle in metric spaces, which is the basic and fundamental result in fixed point theory. Because of its importance, many authors improved, generalized and extended the result either by defining a new contractive mapping in complete metric spaces or by investigating the character of the iterative sequence to provide the existence of fixed points in different ambient spaces, for more details we refer [1, 6, 14, 17, 18].

In 2008, Dutta and Choudhury[11] introduced a new generalization of contraction condition by using altering distance functions and proved the existence of its fixed points in complete metric spaces.

In 2009, Zhang and Song[21] introduced generalized $\varphi-$contraction for a pair of mappings and proved the existence of its common fixed points. In the same year, with the idea of the results of Dutta and Choudhury[11], Doric[10] established a fixed point theorem which is the generalization of the results of [21], for more details we refer [8, 9].

In 2012, Samet, Vetro and Vetro [20] introduced the notion of $\alpha-\psi-$contractive and $\alpha-$admissible mappings and proved the fixed point theorems in complete metric spaces. Further, using the notion of

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$\alpha-$admissible mappings many authors extended it to a pair of mappings and generalized many known fixed point theorems including the Banach contraction principle, for more details we refer [12, 13, 15, 16, 19].

In 2017, Ansari, Dolicanin-Djekic, Dosenovic and Radenovic [3] introduced new functions and using the concept of $\alpha-$admissible and $\mu-$subadmissible mappings they proved fixed point theorems and coupled coincidence point theorems in metric spaces, for more details we refer [4, 5].

In 2018, Cho [7] introduced the notion of generalized weakly contractive mappings in metric spaces and proved a fixed point theorem for generalized weakly contractive mappings in complete metric spaces, and it generalizes the results of [11, 21].

Throughout this paper, we denote the real line by $\mathbb{R}$, $\mathbb{R}^+ = [0, \infty)$, and $\mathbb{N}$ is the set of all natural numbers.

In this paper, motivated and inspired by the works of Cho [7] and Ansari, Dolicanin-Djekic, Dosenovic and Radenovic [3], we introduce the notion of generalized $F - H - \phi - \psi - \varphi-$weakly contractive mappings in metric spaces and prove the existence of fixed points of generalized $F - H - \phi - \psi - \varphi-$weakly contractive mappings in complete metric spaces.

In Section 2, we present basic definitions, lemmas, theorems that are needed to develop our main results, and we introduce the notion of generalized $F - H - \phi - \psi - \varphi-$weakly contractive mappings in metric spaces. In Section 3, we prove the existence of fixed points of generalized $F-H-\phi-\psi-\varphi-$weakly contractive mappings in complete metric spaces and in Section 4, we draw some corollaries and provide examples to illustrate our main results.

2. Preliminaries

Theorem 2.1 ([10])**.** *Let (X, d) be a complete metric space and $S, T : X \to X$ be two functions such that*

$$\psi(d(Tx, Sy)) \leqslant \psi(M(x, y)) - \phi(M(x, y)),$$

for all $x, y \in X$, where

(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0 \iff t = 0$.
(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a lower semicontinuous function with* $\phi(t) = 0 \iff t = 0$.
(iii) $M(x, y) = \max\{d(x, y), d(Tx, x), d(Sy, y), \frac{1}{2}[d(y, Tx) + d(x, Sy)]\}$.

Then there exists a unique point $u \in X$ such that $Tu = u = Su$.

In 2011, Choudhury, Konar, Rhoades and Metiya [9] introduced the notion of generalized weakly contractive mapping as follows.

Definition 2.2 ([9])**.** Let (X, d) be a metric space, T a self-mapping of X. We shall call T a generalized weakly contractive mapping if for any $x, y \in X$,

$$\psi(d(Tx, Ty)) \leqslant \psi(m(x, y)) - \phi(\max\{d(x, y), d(y, Ty)\}),$$

where

(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous monotone increasing function with $\psi(t) = 0 \iff t = 0$.
(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $\phi(t) = 0 \iff t = 0$.
(iii) $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$.

Theorem 2.3 ([9])**.** *Let (X, d) be a complete metric space, T a generalized weakly contractive self-mapping of X. Then T has a unique fixed point.*

Definition 2.4 ([20])**.** Let T be a self mapping on X and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. We say that T is $\alpha-$admissible mapping if for any $x, y \in X$ with $\alpha(x, y) \geqslant 1 \implies \alpha(Tx, Ty) \geqslant 1$.

Definition 2.5 ([19]). Let T be a self mapping on X and let $\mu : X \times X \to \mathbb{R}^+$ be a function. We say that T is $\mu-$subadmissible mapping if for any $x, y \in X$ with $\mu(x, y) \leqslant 1 \implies \mu(Tx, Ty) \leqslant 1$.

In 2014, Ansari [2] introduced the concept of $C-$class functions and many authors proved the generalizations of many important results in fixed point theory under the consideration of $C-$class function as a main source.

Definition 2.6 ([2]). A mapping $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a $C-$class function if it is continuous and for any $s, t \in \mathbb{R}^+$ the function G satisfies the following conditions:

(i) $G(s, t) \leqslant s$.
(ii) $G(s, t) = s$ implies that either $s = 0$ or $t = 0$.

The family of all $C-$class functions is denoted by ζ.

The following functions belong to ζ.

(i) $G(s, t) = s - t$ for any $s, t \in \mathbb{R}^+$.
(ii) $G(s, t) = ks$ for any $s, t \in \mathbb{R}^+$ where $0 < k < 1$.
(iii) $G(s, t) = \frac{s}{(1+t)^r}$ for any $s, t \in \mathbb{R}^+$ where $r \in \mathbb{R}^+$.
(iv) $G(s, t) = s\beta(s)$ for any $s, t \in \mathbb{R}^+$ where $\beta : \mathbb{R}^+ \to [0, 1)$ is continuous.
(v) $G(s, t) = s - \phi(s)$ for any $s, t \in \mathbb{R}^+$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\phi(t) = 0$ if and only if $t = 0$.
(vi) $G(s, t) = sh(s, t)$ for any $s, t \in \mathbb{R}^+$ where $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous such that $h(s, t) < 1$ for any $s, t \in \mathbb{R}^+$.

Definition 2.7 ([3]). A function $H : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type I if it is continuous and $x \geqslant 1 \implies H(1, y) \leqslant H(x, y)$ for any $x \in \mathbb{R}, y \in \mathbb{R}^+$.

The following are the examples of function of subclass of type I for any $x \in \mathbb{R}, y \in \mathbb{R}^+$:

(i) $H(x, y) = (y + l)^x, l > 1$.
(ii) $H(x, y) = (x + l)^y, l > 1$.
(iii) $H(x, y) = xy^n$.
(iv) $H(x, y) = xy$.
(v) $H(x, y) = y$.

Definition 2.8 ([3]). Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a mapping. We say that the pair (F, H) is a upclass of type I if F is continuous, H is a function of subclass of type I and satisfies

(i) $0 \leqslant x \leqslant 1 \implies F(x, y) \leqslant F(1, y)$.
(ii) $H(1, y_1) \leqslant F(x, y_2) \implies y_1 \leqslant xy_2$ for any $x, y, y_1, y_2 \in \mathbb{R}^+$.

The following are the examples of function of upper class of type I for any $x \in \mathbb{R}, y, s, t \in \mathbb{R}^+$:

(i) $H(x, y) = (y + l)^x, l > 1, F(s, t) = st + l$.
(ii) $H(x, y) = (x + l)^y, l > 1, F(s, t) = (1 + l)^{st}$.
(iii) $H(x, y) = xy^n, F(s, t) = s^n t^n$.
(iv) $H(x, y) = xy, F(s, t) = st$.
(v) $H(x, y) = y, F(s, t) = st$.

Definition 2.9 ([3]). Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a mapping. We say that the pair (F, H) is a special upclass of type I if F is continuous, H is a function of subclass of type I and satisfies :

(i) $0 \leqslant s \leqslant 1 \implies F(s, t) \leqslant F(1, t)$.
(ii) $H(1, y) \leqslant F(1, t) \implies y \leqslant t$ for any $y, s, t \in \mathbb{R}^+$.

The following are the examples of function of special upclass of type I for any $x \in \mathbb{R}, y, s, t \in \mathbb{R}^+$:

(i) $H(x,y) = (y^k + l)^{x^n}, l > 1, F(s,t) = s^m t^k + l.$
(ii) $H(x,y) = (x^n + l)^{y^k}, l > 1, F(s,t) = (1 + l)^{s^m t^k}.$
(iii) $H(x,y) = x^n y^k, F(s,t) = s^p t^k.$
(iv) $H(x,y) = xy, F(s,t) = st.$
(v) $H(x,y) = y, F(s,t) = st.$

Remark 2.10 ([3]). Each pair (F,H) of upclass of type I is a pair (F,H) of special upclass of type I, but its converse is not true.

Definition 2.11 ([7]). Let (X,d) be a metric space, T a self-mapping of X. Then T is called a generalized weakly contractive mapping in the sense of Cho, if for any $x, y \in X$,

$$\psi(d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)) \leqslant \psi(m(x,y,d,T,\varphi)) - \phi(l(x,y,d,T,\varphi)),$$

where

(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and $\psi(t) = 0 \iff t = 0$.
(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous function and $\phi(t) = 0 \iff t = 0$.
(iii) $m(x,y,d,T,\varphi) = \max\{d(x,y) + \varphi(x) + \varphi(y), d(x,Tx) + \varphi(x) + \varphi(Tx), d(y,Ty) + \varphi(y) + \varphi(Ty), \frac{1}{2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Tx) + \varphi(y) + \varphi(Tx)]\}$.
(iv) $l(x,y,d,T,\varphi) = \max\{d(x,y) + \varphi(x) + \varphi(y), d(y,Ty) + \varphi(y) + \varphi(Ty)\}$.
(v) $\varphi : X \to \mathbb{R}^+$ is a lower semicontinuous function.

Theorem 2.12 ([7]). *Let X be a complete metric space. If T is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.*

We denote $\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+ \mid \psi \text{ is continuous and } \psi(t) = 0 \iff t = 0\}$.

Based on the results of [2, 7] and new functions of [3], we introduce the notion of generalized $F - H - \phi - \psi - \varphi-$weakly contractive mappings in metric spaces as follows.

Definition 2.13. Let (X,d) be a metric space. Let G be a $C-$class function such that $G(\mathbb{R}^+, \mathbb{R}^+) \subseteq \mathbb{R}^+$. Let $T : X \to X$ be a function. If there exist $\alpha, \mu : X \times X \to \mathbb{R}^+$, $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $H : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$H(\alpha(x,Tx)\alpha(y,Ty), \psi(d(Tx,Ty) + \varphi(Tx) + \varphi(Ty))) \leqslant F(\mu(x,Tx)\mu(y,Ty), G(\psi(M(x,y)), \phi(N(x,y)))), \tag{2.1}$$

for any $x, y \in X$, where $\psi, \phi \in \Psi$, $\varphi : X \to \mathbb{R}^+$ is lower semicontinuous function,

$$M(x,y) = \max\{d(x,y) + \varphi(x) + \varphi(y), d(x,Tx) + \varphi(x) + \varphi(Tx), d(y,Ty) + \varphi(y) + \varphi(Ty), \frac{1}{2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Tx) + \varphi(y) + \varphi(Tx)]\},$$

and
$N(x,y) = \max\{d(x,y) + \varphi(x) + \varphi(y), d(y,Ty) + \varphi(y) + \varphi(Ty)\}$, then we call T is a generalized $F - H - \phi - \psi - \varphi-$weakly contractive mapping.

Example 2.14. Let $X = [0,2]$ with usual metric.
We define $H : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $F, G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $H(x,y) = \frac{xy}{2}, F(s,t) = st$ and

$$G(s,t) = \begin{cases} s - t, & \text{if } s \geqslant t, \\ 0, & \text{otherwise}, \end{cases}$$

for any $x \in \mathbb{R}, y, s, t \in \mathbb{R}^+$.

We define $\varphi : X \to \mathbb{R}^+$ by

$$\varphi(x) = \begin{cases} x, & \text{if } 0 \leqslant x < 1, \\ \frac{x}{4}, & \text{if } x \geqslant 1, \end{cases}$$

for any $x \in X$. Clearly φ is lower semicontinuous.

We define $T : X \to X,\ \ \alpha, \mu : X \times X \to \mathbb{R}^+$ by $T(x) = \frac{x^2}{6-2x}$,

$$\alpha(x,y) = \begin{cases} \frac{1}{2}, & \text{if } x \geqslant y, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mu(x,y) = \begin{cases} \sqrt{2}, & \text{if } x \geqslant y, \\ 2, & \text{otherwise,} \end{cases}$$

for any $x, y \in X$.

We define $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = t$ and $\phi(t) = \frac{t}{t+4}$ for any $t \in \mathbb{R}^+$. Clearly $\psi, \phi \in \Psi$. Without loss of generality, we assume that $x \geqslant y$. We consider

$$d(Tx,Ty) + \varphi(Tx) + \varphi(Ty) \leqslant d(Tx,Ty) + Tx + Ty = 2\ \ Tx = 2\ \ \frac{x^2}{6-2x} = \frac{x^2}{3-x},$$

and hence $\psi(d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)) \leqslant \psi(\frac{x^2}{3-x}) = \frac{x^2}{3-x}$. Since $x, y \in [0,2]$ we have $x \geqslant Tx$ and $y \geqslant Ty$. Therefore

$$\alpha(x,Tx)\alpha(y,Ty)\psi(d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)) \leqslant \frac{x^2}{12-4x},$$

and hence

$$H(\alpha(x,Tx)\alpha(y,Ty), \psi(d(Tx,Ty) + \varphi(Tx) + \varphi(Ty))) \leqslant \frac{x^2}{24-8x}. \tag{2.2}$$

We consider

$$\begin{aligned} M(x,y) &= \max\{d(x,y) + \varphi(x) + \varphi(y), d(x,Tx) + \varphi(x) + \varphi(Tx), d(y,Ty) + \varphi(y) + \varphi(Ty), \\ &\quad \frac{1}{2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Tx) + \varphi(y) + \varphi(Tx)]\} \\ &\geqslant d(x,Tx) + \varphi(x) + \varphi(Tx) \\ &\geqslant \frac{d(x,Tx)}{4} + \frac{x}{4} + \frac{Tx}{4} \\ &= \frac{x}{2}. \end{aligned}$$

Therefore

$$\psi(M(x,y)) \geqslant \psi(\frac{x}{2}) = \frac{x}{2}. \tag{2.3}$$

We consider

$$\begin{aligned} N(x,y) &= \max\{d(x,y) + \varphi(x) + \varphi(y), d(y,Ty) + \varphi(y) + \varphi(Ty)\} \\ &\leqslant \max\{d(x,y) + x + y,\ \ d(y,Ty) + y + Ty\} \\ &= \max\{2x, 2y\} \\ &= 2x. \end{aligned}$$

Therefore
$$\phi(N(x,y)) \leqslant \phi(2x) = \frac{2x}{2x+4}. \tag{2.4}$$
From (2.3) and (2.4), we get
$$\psi(M(x,y)) - \phi(N(x,y)) \geqslant \frac{x}{2} - \frac{2x}{2x+4} = \frac{x^2}{2x+4}.$$
We consider
$$\begin{aligned}
&F(\mu(x,Tx)\mu(y,Ty), G(\psi(M(x,y)), \phi(N(x,y))))\\
&\qquad = F(\mu(x,Tx)\mu(y,Ty), \psi(M(x,y)) - \phi(N(x,y)))\\
&\qquad \text{(since} \quad \psi(M(x,y)) \geqslant \frac{x}{2} \geqslant \frac{2x}{2x+4} \geqslant \phi(N(x,y)))\text{)}\\
&\qquad = \mu(x,Tx)\mu(y,Ty)(\psi(M(x,y)) - \phi(N(x,y)))\\
&\qquad \geqslant 2\ \frac{x^2}{2x+4} = \frac{x^2}{x+2}\\
&\qquad \geqslant \frac{x^2}{24-8x} \qquad\qquad \text{(since} \quad x \in [0,2]\text{)}\\
&\qquad \geqslant H(\alpha(x,Tx)\alpha(y,Ty), \psi(d(Tx,Ty) + \varphi(Tx) + \varphi(Ty))).
\end{aligned}$$
Therefore the inequality (2.1) is satisfied.

Remark 2.15. In Example 2.14, $H(1,y) = \frac{y}{2}, F(1,t) = t$. We observe that if $H(1,y) \leqslant F(1,t)$ then $y \leqslant 2t$ for any $y, t \in \mathbb{R}^+$ and hence the pair (F,H) is not a special upclass of type I.

The following proposition and lemma are useful in the subsequent discussion in proving our main results.

Proposition 2.16. If $\{a_n\}$ and $\{b_n\}$ are two real sequences, $\{b_n\}$ is bounded, then
$$\liminf(a_n + b_n) \leqslant \liminf a_n + \limsup b_n.$$

Lemma 2.17 ([6]). *Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geqslant \epsilon, d(x_{m_k-1}, x_{n_k}) < \epsilon$ and*

(i) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$.
(ii) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$.
(iii) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$.

3. Existence of fixed points

Theorem 3.1. *Let (X,d) be a complete metric space. Let $T : X \to X$ be a function such that*

(i) *T is a generalized $F - H - \phi - \psi - \varphi-$weakly contractive mapping.*
(ii) *T is an $\alpha-$admissible and $\mu-$subadmissible mapping.*
(iii) *the pair (F,H) is a special uplcass of type I.*
(iv) *if $\{x_n\}$ is any sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1, \mu(x_n, x_{n+1}) \leqslant 1$ and $\{x_n\} \to z$ for any $n \in \mathbb{N}\cup\{0\}$ then $\alpha(z,Tz) \geqslant 1$ and $\mu(z,Tz) \leqslant 1$.*

Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\mu(x_0, Tx_0) \leqslant 1$. Then there exists $u \in X$ such that $Tu = u$ and $\varphi(u) = 0$. Further, if there exists $y_0 \in X$ such that $\alpha(y_0, Ty_0) \geqslant 1$ and $\mu(y_0, Ty_0) \leqslant 1$. Then there exists $v \in X$ such that $Tv = v$ and $\varphi(v) = 0$. In this case, $v = u$, that is the fixed point u is unique in this sense.

Proof. Let $x_0 \in X$. We define a sequence $\{x_n\}$ in X such that $Tx_n = x_{n+1}$, for any $n \in \mathbb{N} \cup \{0\}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x_n = x_{n+1} = Tx_n$. Therefore T has a fixed point. Assume that $x_n \neq x_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$. From our assumption, we have $\alpha(x_0, x_1) \geqslant 1$ and $\mu(x_0, x_1) \leqslant 1$. Since T is an $\alpha-$admissible and $\mu-$subadmissible mapping, we have $\alpha(Tx_0, Tx_1) \geqslant 1$ and $\mu(Tx_0, Tx_1) \leqslant 1$, that is $\alpha(x_1, x_2) \geqslant 1$ and $\mu(x_1, x_2) \leqslant 1$. On continuing this process, we get

$$\alpha(x_n, x_{n+1}) \geqslant 1 \text{ and } \mu(x_n, x_{n+1}) \leqslant 1 \quad \text{for any} \ \ n \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

We consider

$$\begin{aligned} &H(1, \psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}))) \\ &\qquad = H(1, \psi(d(Tx_n, Tx_{n+1}) + \varphi(Tx_n) + \varphi(Tx_{n+1}))) \\ &\qquad \leqslant H(\alpha(x_n, Tx_n)\alpha(x_{n+1}, Tx_{n+1}), \psi(d(Tx_n, Tx_{n+1}) + \varphi(Tx_n) + \varphi(Tx_{n+1}))) \\ &\qquad \leqslant F(\mu(x_n, Tx_n)\mu(x_{n+1}, Tx_{n+1}), G(\psi(M(x_n, x_{n+1})), \phi(N(x_n, x_{n+1})))) \\ &\qquad \leqslant F(1, G(\psi(M(x_n, x_{n+1})), \phi(N(x_n, x_{n+1})))). \end{aligned}$$

This imples that

$$\psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})) \leqslant G(\psi(M(x_n, x_{n+1})), \phi(N(x_n, x_{n+1}))). \tag{3.2}$$

We consider

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_n, Tx_n) + \varphi(x_n) + \varphi(Tx_n), \\ &\qquad d(x_{n+1}, Tx_{n+1}) + \varphi(x_{n+1}) + \varphi(Tx_{n+1}), \\ &\qquad \frac{1}{2}[d(x_n, Tx_{n+1}) + \varphi(x_n) + \varphi(Tx_{n+1}) + d(x_{n+1}, Tx_n) + \varphi(x_{n+1}) + \varphi(Tx_n)]\} \\ &\leqslant \max\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}), \\ &\qquad \frac{1}{2}[d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})]\} \\ &= \max\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})\}, \end{aligned}$$

and

$$\begin{aligned} N(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_{n+1}, Tx_{n+1}) + \varphi(x_{n+1}) + \varphi(Tx_{n+1})\} \\ &= \max\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), \ d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})\}. \end{aligned}$$

Let us suppose $d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) < d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})$. Then

$$M(x_n, x_{n+1}) = N(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}).$$

From (3.2) we have

$$\begin{aligned} &\psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})) \\ &\qquad \leqslant G(\psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})), \phi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}))) \\ &\qquad \leqslant \psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})). \end{aligned}$$

Therefore

$$\begin{aligned} &G(\psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})), \phi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}))) \\ &\qquad = \psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})). \end{aligned}$$

This implies that either $\psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})) = 0$ or $\phi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})) = 0$. Therefore $d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}) = 0$. Hence $x_{n+1} = x_{n+2}$ and $\varphi(x_{n+1}) =$

$\varphi(x_{n+2}) = 0$, a contradiction. Therefore $M(x_n, x_{n+1}) = N(x_n, x_{n+1}) = d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})$. Let $d_n = d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})$. Then $d_n \geqslant d_{n+1}$ and hence the sequence $\{d_n\}$ is a decreasing sequence. Let $\lim_{n\to\infty} d_n = r$. From (3.2) we have $\psi(d_{n+1}) \leqslant G(\psi(d_n), \phi(d_n))$. On applying limits as $n \to \infty$, we get $\psi(r) \leqslant G(\psi(r), \phi(r)) \leqslant \psi(r)$ and hence $G(\psi(r), \phi(r)) = \psi(r)$. This implies that either $\psi(r) = 0$ or $\phi(r) = 0$ and hence $r = 0$. Therefore

$$\begin{cases} \lim_{n\to\infty} d_n = \lim_{n\to\infty} [d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})] = 0. \\ \text{That is,} \quad \lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \quad \text{and} \quad \lim_{n\to\infty} \varphi(x_n) = 0. \end{cases} \tag{3.3}$$

We now show that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose that the sequence $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geqslant \epsilon$ and $d(x_{m_k-1}, x_{n_k}) < \epsilon$. By Lemma 2.17 we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{3.4}$$

By using the triangle inequality we have

$$\epsilon \leqslant d(x_{m_k}, x_{n_k}) \leqslant d(x_{m_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}).$$

Now by applying Proposition 2.16 with $a_k = d(x_{m_k}, x_{n_k+1})$ and $b_k = d(x_{n_k+1}, x_{n_k})$ we have

$$\epsilon \leqslant \liminf_{k\to\infty} d(x_{m_k}, x_{n_k+1}). \tag{3.5}$$

Now by applying the triangle inequality we have

$$d(x_{m_k}, x_{n_k+1}) \leqslant d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}).$$

On applying limit superior as $k \to \infty$ we get

$$\limsup_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leqslant \epsilon. \tag{3.6}$$

From (3.5) and (3.6) we get

$$\epsilon \leqslant \liminf_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leqslant \limsup_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leqslant \epsilon.$$

Therefore

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \epsilon. \tag{3.7}$$

By applying the triangle inequality we have

$$\epsilon \leqslant d(x_{m_k}, x_{n_k}) \leqslant d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k}).$$

Now by applying Proposition 2.16 with $a_k = d(x_{m_k+1}, x_{n_k})$ and $b_k = d(x_{m_k}, x_{m_k+1})$ we have

$$\epsilon \leqslant \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k}). \tag{3.8}$$

By the triangle inequality we have

$$d(x_{m_k+1}, x_{n_k}) \leqslant d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{n_k}).$$

On applying limit superior as $k \to \infty$ we get

$$\limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leqslant \epsilon. \tag{3.9}$$

From (3.8) and (3.9) we get

$$\epsilon \leqslant \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leqslant \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leqslant \epsilon.$$

Therefore

$$\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k}) = \epsilon. \tag{3.10}$$

We have

$$\epsilon \leqslant d(x_{m_k}, x_{n_k}) \leqslant d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}).$$

Now by applying Proposition 2.16 with $a_k = d(x_{m_k+1}, x_{n_k+1})$ and $b_k = d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k})$ we have

$$\epsilon \leqslant \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}). \tag{3.11}$$

Also, we have

$$d(x_{m_k+1}, x_{n_k+1}) \leqslant d(x_{m_k+1}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}).$$

On applying limit superior as $k \to \infty$ we get

$$\limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leqslant \epsilon. \tag{3.12}$$

From (3.11) and (3.12) we get

$$\epsilon \leqslant \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leqslant \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leqslant \epsilon.$$

Therefore

$$\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \epsilon. \tag{3.13}$$

We consider

$$\begin{aligned}
&H(1, \psi(d(x_{m_k+1}, x_{n_k+1}) + \varphi(x_{m_k+1}) + \varphi(x_{n_k+1}))\\
&\qquad = H(1, \psi(d(Tx_{m_k}, Tx_{n_k}) + \varphi(Tx_{m_k}) + \varphi(Tx_{n_k}))\\
&\qquad \leqslant H(\alpha(x_{m_k}, Tx_{m_k})\alpha(x_{n_k}, Tx_{n_k}), \psi(d(Tx_{m_k}, Tx_{n_k}) + \varphi(Tx_{m_k}) + \varphi(Tx_{n_k})))\\
&\qquad \leqslant F(\mu(x_{m_k}, Tx_{m_k})\mu(x_{n_k}, Tx_{n_k}), G(\psi(M(x_{m_k}, x_{n_k})), \phi(N(x_{m_k}, x_{n_k}))))\\
&\qquad \leqslant F(1, G(\psi(M(x_{m_k}, x_{n_k})), \phi(N(x_{m_k}, x_{n_k})))).
\end{aligned}$$

This imples that

$$\begin{aligned}
&\psi(d(x_{m_k+1}, x_{n_k+1}) + \varphi(x_{m_k+1}) + \varphi(x_{n_k+1}))\\
&\qquad \leqslant G(\psi(M(x_{m_k}, x_{n_k}), \phi(N(x_{m_k}, x_{n_k})))).
\end{aligned} \tag{3.14}$$

We now consider

$$\begin{aligned}
M(x_{m_k}, x_{n_k}) &= \max\{d(x_{m_k}, x_{n_k}) + \varphi(x_{m_k}) + \varphi(x_{n_k}), d(x_{m_k}, Tx_{m_k}) + \varphi(x_{m_k}) + \varphi(Tx_{m_k}),\\
&\quad d(x_{n_k}, Tx_{n_k}) + \varphi(x_{n_k}) + \varphi(Tx_{n_k}),\\
&\quad \frac{1}{2}[d(x_{m_k}, Tx_{n_k}) + \varphi(x_{m_k}) + \varphi(Tx_{n_k}) + d(x_{n_k}, Tx_{m_k}) + \varphi(x_{n_k}) + \varphi(Tx_{m_k})]\}\\
&= \max\{d(x_{m_k}, x_{n_k}) + \varphi(x_{m_k}) + \varphi(x_{n_k}), d(x_{m_k}, x_{m_k+1}) + \varphi(x_{m_k}) + \varphi(x_{m_k+1}),\\
&\quad d(x_{n_k}, x_{n_k+1}) + \varphi(x_{n_k}) + \varphi(x_{n_k+1}),\\
&\quad \frac{1}{2}[d(x_{m_k}, x_{n_k+1}) + \varphi(x_{m_k}) + \varphi(x_{n_k+1}) + d(x_{n_k}, x_{m_k+1}) + \varphi(x_{n_k}) + \varphi(x_{m_k+1})]\},
\end{aligned}$$

and

$$\begin{aligned} N(x_{m_k}, x_{n_k}) &= \max\{d(x_{m_k}, x_{n_k}) + \varphi(x_{m_k}) + \varphi(x_{n_k}), d(x_{n_k}, Tx_{n_k}) + \varphi(x_{n_k}) + \varphi(Tx_{n_k})\} \\ &= \max\{d(x_{m_k}, x_{n_k}) + \varphi(x_{m_k}) + \varphi(x_{n_k}), d(x_{n_k}, x_{n_k+1}) + \varphi(x_{n_k}) + \varphi(x_{n_k+1})\}. \end{aligned}$$

On applying limits as $k \to \infty$, we get

$$\lim_{k\to\infty} M(x_{m_k}, x_{n_k}) = \epsilon = \lim_{k\to\infty} N(x_{m_k}, x_{n_k}).$$

On applying limits as $k \to \infty$ to the equation (3.14), we get

$$\psi(\epsilon) \leqslant G(\psi(\epsilon), \phi(\epsilon)) \leqslant \psi(\epsilon)$$

and hence $\epsilon = 0$, a contradiction. Therefore the sequence $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. From (iv), we have $\alpha(u, Tu) \geqslant 1$ and $\mu(u, Tu) \leqslant 1$.

Since φ is a lower semicontiuous function, we have $\varphi(u) \leqslant \lim_{n\to\infty} \inf \varphi(x_n) = 0$ and hence $\varphi(u) = 0$. We now show that u is a fixed point of T. We consider

$$\begin{aligned} M(x_n, u) = \max\{&d(x_n, u) + \varphi(x_n) + \varphi(u), d(x_n, Tx_n) + \varphi(x_n) + \varphi(Tx_n), d(u, Tu) + \varphi(u) + \varphi(Tu), \\ &\frac{1}{2}[d(x_n, Tu) + \varphi(x_n) + \varphi(Tu) + d(u, Tx_n) + \varphi(u) + \varphi(Tx_n)]\} \\ = \max\{&d(x_n, u) + \varphi(x_n) + \varphi(u), d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(u, Tu) + \varphi(u) + \varphi(Tu), \\ &\frac{1}{2}[d(x_n, Tu) + \varphi(x_n) + \varphi(Tu) + d(u, x_{n+1}) + \varphi(u) + \varphi(x_{n+1})]\}, \end{aligned}$$

and

$$N(x_n, u) = \max\{d(x_n, u) + \varphi(x_n) + \varphi(u), d(u, Tu) + \varphi(u) + \varphi(Tu)\}.$$

On applying limits as $n \to \infty$, we get

$$\lim_{n\to\infty} M(x_n, u) = d(u, Tu) + \varphi(Tu) = \lim_{n\to\infty} N(x_n, u).$$

We now consider

$$\begin{aligned} &H(1, \psi(d(x_{n+1}, Tu) + \varphi(x_{n+1}) + \varphi(Tu))) \\ &\qquad = H(1, \psi(d(Tx_n, Tu) + \varphi(Tx_n) + \varphi(Tu))) \\ &\qquad \leqslant H(\alpha(x_n, Tx_n)\alpha(u, Tu), \psi(d(Tx_n, Tu) + \varphi(Tx_n) + \varphi(Tu))) \\ &\qquad \leqslant F(\mu(x_n, Tx_n)\mu(u, Tu), G(\psi(M(x_n, u)), \phi(N(x_n, u)))) \\ &\qquad \leqslant F(1, G(\psi(M(x_n, u)), \phi(N(x_n, u)))). \end{aligned}$$

This imples that

$$\psi(d(x_{n+1}, Tu) + \varphi(x_{n+1}) + \varphi(Tu)) \leqslant G(\psi(M(x_n, u)), \phi(N(x_n, u))). \tag{3.15}$$

On applying limits as $n \to \infty$, we get

$$\psi(d(u, Tu) + \varphi(Tu)) \leqslant G(\psi(d(u, Tu) + \varphi(Tu)), \phi(d(u, Tu) + \varphi(Tu))) \leqslant \psi(d(u, Tu) + \varphi(Tu)).$$

From the definition of $C-$class function, we get $\psi(d(u, Tu) + \varphi(Tu)) = 0$ or $\phi(d(u, Tu) + \varphi(Tu)) = 0$ and hence $d(u, Tu) = \varphi(Tu) = 0$. Therefore $Tu = u$ and $\varphi(u) = 0$. Now, if $y_0 \in X$ is such that $\alpha(y_0, Ty_0) \geqslant 1$

and $\mu(y_0, Ty_0) \leqslant 1$ then by the above argument, it follows that there exists $v \in X$ such that $Tv = v$ and $\alpha(v, Tv) \geqslant 1, \mu(v, Tv) \leqslant 1$, and $\varphi(v) = 0$. We now show that $v = u$. We consider

$$\begin{aligned} H(1, \psi(d(u,v))) &= H(1, \psi(d(u,v) + \varphi(u) + \varphi(v))) \\ &= H(1, \psi(d(Tu, Tv) + \varphi(Tu) + \varphi(Tv))) \\ &\leqslant H(\alpha(u, Tu)\alpha(v, Tv), \psi(d(Tu, Tv) + \varphi(Tu) + \varphi(Tv))) \\ &\leqslant F(\mu(u, Tu)\mu(v, Tv), G(\psi(M(u,v)), \phi(N(u,v)))) \\ &\leqslant F(1, G(\psi(M(u,v)), \phi(N(u,v)))). \end{aligned}$$

Therefore $\psi(d(u,v)) \leqslant G(\psi(M(u,v)), \phi(N(u,v))) = G(\psi(d(u,v)), \phi(d(u,v))) \leqslant \psi(d(u,v))$.

Hence $G(\psi(d(u,v)), \phi(d(u,v))) = \psi(d(u,v))$. From the definition of $C-$ class function, we get either $\psi(d(u,v)) = 0$ or $\phi(d(u,v)) = 0$ and hence $v = u$. Therefore T has a unique fixed point $u \in X$ and $\varphi(u) = 0$. □

4. Corollaries and examples

Corollary 4.1. *Let (X, d) be a complete metric space. Let $G(\mathbb{R}^+, \mathbb{R}^+) \subseteq \mathbb{R}^+$. Let $T : X \to X$ be a function. Assume that*

(i) *there exist $\alpha, \mu : X \times X \to \mathbb{R}^+$ and $\psi, \phi \in \Psi$ such that*

$$[\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) + l]^{\alpha(x,Tx)\alpha(y,Ty)} \leqslant \mu(x, Tx)\mu(y, Ty)G(\psi(M(x,y)), \phi(N(x,y))) + l$$

for any $x, y \in X$, $l > 1$, where $\varphi : X \to \mathbb{R}^+$ is lower semicontinuous,

$$\begin{aligned} M(x,y) = \max\{&d(x,y) + \varphi(x) + \varphi(y), d(x, Tx) + \varphi(x) + \varphi(Tx), d(y, Ty) + \varphi(y) + \varphi(Ty), \\ &\frac{1}{2}[d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)]\}, \\ N(x,y) = \max\{&d(x,y) + \varphi(x) + \varphi(y), d(y, Ty) + \varphi(y) + \varphi(Ty)\}. \end{aligned}$$

(ii) *T is an $\alpha-$admissible and $\mu-$subadmissible mapping.*

(iii) *(F, H) is a special uplcass of type I.*

(iv) *if $\{x_n\}$ is any sequence in X such that $\alpha(x_n, Tx_n) \geqslant 1, \mu(x_n, Tx_n) \geqslant 1$ and $x_n \to z$ then $\alpha(z, Tz) \geqslant 1$ and $\mu(z, Tz) \leqslant 1$.*

Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\mu(x_0, Tx_0) \leqslant 1$. Then there exists $u \in X$ such that $Tu = u$ and $\varphi(u) = 0$. Furthermore, if there exists $y_0 \in X$ such that $\alpha(y_0, Ty_0) \geqslant 1$ and $\mu(y_0, Ty_0) \leqslant 1$ then there exists $v \in X$ such that $Tv = v$ and $\varphi(v) = 0$. In this case, $v = u$, that is the fixed point u is unique in this sense.

Proof. Follows by choosing $H(x,y) = (y + l)^x$ and $F(s,t) = st + l$ for any $x \in \mathbb{R}$, $y, s, t \in \mathbb{R}^+$ in Theorem 3.1. □

If $H(x,y) = xy$, $F(s,t) = st$ for all $x \in \mathbb{R}, y, s, t \in \mathbb{R}^+, \mu(x,y) = 1, \varphi(x) = 0$ for all $x, y \in X$ in Theorem 3.1, then we obtain the following.

Corollary 4.2. *Let (X, d) be a complete metric space. Let $G(\mathbb{R}^+, \mathbb{R}^+) \subseteq \mathbb{R}^+$. Let $T : X \to X$ be a function. Assume that*

(i) *there exist $\alpha : X \times X \to \mathbb{R}^+$ and $\psi, \phi \in \Psi$ such that*

$$\alpha(x, Tx)\alpha(y, Ty)\psi(d(Tx, Ty)) \leqslant G(\psi(M(x,y)), \phi(N(x,y))),$$

for any $x, y \in X$, where

$$M(x,y) = \max\{d(x,y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$$

and

$$N(x,y) = \max\{d(x,y), d(y, Ty)\}.$$

(ii) T *is an* α*−admissible mapping.*
(iii) *if* $\{x_n\}$ *is any sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geqslant 1$ *and* $\{x_n\} \to z$ *then* $\alpha(z, Tz) \geqslant 1$.

Assume that there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$. *Then there exists* $u \in X$ *such that* $Tu = u$. *Furthermore, if there exists* $y_0 \in X$ *such that* $\alpha(y_0, Ty_0) \geqslant 1$ *and* $\mu(y_0, Ty_0) \leqslant 1$ *then there exists* $v \in X$ *such that* $Tv = v$ *and* $\varphi(v) = 0$. *In this case,* $v = u$, *that is the fixed point* u *is unique in this sense.*

Corollary 4.3. *Let* (X, d) *be a complete metric space. Let* $T : X \to X$ *be a function. Assume that there exist* $\psi, \phi \in \Psi$ *such that* $\psi(t) \geqslant \phi(s)$ *whenever* $t \geqslant s$ *and*

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leqslant \psi(M(x, y)) - \phi(N(x, y)),$$

for any $x, y \in X$, *where* $\varphi : X \to \mathbb{R}^+$ *is lower semicontinuous,*

$$\begin{aligned} M(x, y) = \max\{&d(x, y) + \varphi(x) + \varphi(y), d(x, Tx) + \varphi(x) + \varphi(Tx), d(y, Ty) + \varphi(y) + \varphi(Ty), \\ &\frac{1}{2}[d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)]\}, \\ N(x, y) = \max\{&d(x, y) + \varphi(x) + \varphi(y), d(y, Ty) + \varphi(y) + \varphi(Ty)\}. \end{aligned}$$

Let $x_0 \in X$. *Then there exists a unique* $u \in X$ *such that* $Tu = u$ *and* $\varphi(u) = 0$.

Proof. Follows by choosing

$$G(s, t) = \begin{cases} s - t, & \text{if } s \geqslant t, \\ 0, & \text{otherwise,} \end{cases}$$

$H(x, y) = xy, \quad F(s, t) = st$, for all $x \in \mathbb{R}, y, s, t \in \mathbb{R}^+$ and $\alpha(x, y) = 1 = \mu(x, y)$, for all $x, y \in X$ in Theorem 3.1. □

The proof of the following corollary is similar to that of Corollary 4.3.

Corollary 4.4. *Let* (X, d) *be a complete metric space. Let* $T : X \to X$ *be a function. Assume that there exist* $\phi, \psi \in \Psi$ *such that*

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leqslant \psi(M(x, y)) - \phi(M(x, y))$$

for any $x, y \in X$, *where* $\varphi : X \to \mathbb{R}^+$ *is lower semicontinuous,*

$$\begin{aligned} M(x, y) = \max\{&d(x, y) + \varphi(x) + \varphi(y), d(x, Tx) + \varphi(x) + \varphi(Tx), d(y, Ty) + \varphi(y) + \varphi(Ty), \\ &\frac{1}{2}[d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)]\}. \end{aligned}$$

Let $x_0 \in X$. *Then there exists a unique* $u \in X$ *such that* $Tu = u$ *and* $\varphi(u) = 0$.

If $\varphi = 0$ in Corollary 4.3 then we obtain the following.

Corollary 4.5. *Let* (X, d) *be a complete metric space. Let* $T : X \to X$ *be a function. Assume that there exist* $\psi, \phi \in \Psi$ *such that* $\psi(t) \geqslant \phi(s)$ *for any* $t \geqslant s$ *and*

$$\psi(d(Tx, Ty)) \leqslant \psi(M(x, y)) - \phi(N(x, y)),$$

for any $x, y \in X$, *where*

$$\begin{aligned} M(x, y) &= \max\{d(x, y), d(x, Tx)), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}, \\ N(x, y) &= \max\{d(x, y), d(y, Ty)\}. \end{aligned}$$

Let $x_0 \in X$. *Then there exists a unique* $u \in X$ *such that* $Tu = u$.

We now present an example in support of Theorem 3.1.

Example 4.6. Let $X = [0, 1]$ with usual metric. We define $H : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $F, G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $H(x, y) = xy$, $F(s, t) = st$ and

$$G(s,t) = \begin{cases} s - t, & \text{if } s \geqslant t, \\ 0, & \text{otherwise,} \end{cases}$$

for any $x \in \mathbb{R}$, $y, s, t \in \mathbb{R}^+$. We define $\varphi : X \to \mathbb{R}^+$ by

$$\varphi(x) = \begin{cases} x, & \text{if } 0 \leqslant x < \frac{1}{2}, \\ \frac{x}{2}, & \text{if } x \geqslant \frac{1}{2}, \end{cases}$$

for any $x \in X$. Clearly φ is lower semicontinuous. We define $T : X \to X$, $\alpha, \mu : X \times X \to \mathbb{R}^+$ by $T(x) = \frac{x^2}{2[8-x]}$,

$$\alpha(x,y) = \begin{cases} \sqrt{2}, & \text{if } x \geqslant y, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mu(x,y) = \begin{cases} \frac{1}{\sqrt{3}}, & \text{if } x \geqslant y, \\ 2, & \text{otherwise,} \end{cases}$$

for any $x, y \in X$. First we show that T is $\alpha-$admissible. Let $x, y \in X$ be such that $\alpha(x, y) \geqslant 1$. Then $x \geqslant y$. Clearly $\frac{x^2}{2[8-x]} \geqslant \frac{y^2}{2[8-y]}$ and hence $Tx \geqslant Ty$. Therefore $\alpha(Tx, Ty) \geqslant 1$ and hence T is $\alpha-$admissible mapping. Similarly we can show that T is $\mu-$subadmissible mapping. We define $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = 18t$ and $\phi(t) = \frac{9t}{1+4t}$ for any $t \in \mathbb{R}^+$. Clearly $\psi, \phi \in \Psi$. Without loss of generality, we assume that $x \geqslant y$. We consider

$$d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \leqslant d(Tx, Ty) + Tx + Ty = 2\ \ Tx = 2\ \ \frac{x^2}{2[8-x]} = \frac{x^2}{8-x},$$

and hence

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leqslant \psi(\frac{x^2}{8-x}) = \frac{18x^2}{8-x}.$$

Since $x, y \in [0, 1]$ we have $x \geqslant Tx$ and $y \geqslant Ty$. Therefore

$$\alpha(x, Tx)\alpha(y, Ty)\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leqslant \frac{36x^2}{8-x}. \tag{4.1}$$

We consider

$$\begin{aligned} M(x,y) &= \max\{d(x,y) + \varphi(x) + \varphi(y), d(x,Tx) + \varphi(x) + \varphi(Tx), d(y,Ty) + \varphi(y) + \varphi(Ty), \\ &\quad \frac{1}{2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Tx) + \varphi(y) + \varphi(Tx)]\} \\ &\geqslant d(x,Tx) + \varphi(x) + \varphi(Tx) \\ &\geqslant \frac{d(x,Tx)}{2} + \frac{x}{2} + \frac{Tx}{2} \\ &= x. \end{aligned}$$

Therefore

$$\psi(M(x,y)) \geqslant \psi(x) = 18x. \tag{4.2}$$

We consider

$$\begin{aligned} N(x,y) &= \max\{d(x,y)+\varphi(x)+\varphi(y), d(y,Ty)+\varphi(y)+\varphi(Ty)\} \\ &\leqslant \max\{d(x,y)+x+y,\ \ d(y,Ty)+y+Ty\} \\ &= \max\{2x,2y\} \\ &= 2x. \end{aligned}$$

Therefore

$$\phi(N(x,y)) \leqslant \phi(2x) = \frac{18x}{1+8x}. \tag{4.3}$$

From (4.2) and (4.3) we get

$$\psi(M(x,y)) - \phi(N(x,y)) \geqslant 18x - \frac{18x}{1+8x} = \frac{144x^2}{1+8x}.$$

We consider

$$\begin{aligned} &F(\mu(x,Tx)\mu(y,Ty), G(\psi(M(x,y)), \phi(N(x,y)))) \\ &\qquad = F(\mu(x,Tx)\mu(y,Ty), \psi(M(x,y)) - \phi(N(x,y))) \\ &\qquad (\text{since} \quad \psi(M(x,y)) \geqslant 18x \geqslant \frac{18x}{1+8x} \geqslant \phi(N(x,y))) \\ &\qquad = \mu(x,Tx)\mu(y,Ty)(\psi(M(x,y)) - \phi(N(x,y))) \\ &\qquad \geqslant \frac{1}{3}\ \frac{144x^2}{1+8x} \\ &\qquad \geqslant \frac{36x^2}{8-x} \qquad (\text{since} \quad x\in[0,1]) \\ &\qquad \geqslant \alpha(x,Tx)\alpha(y,Ty)\psi(d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)) \\ &\qquad = H(\alpha(x,Tx)\alpha(y,Ty), \psi(d(Tx,Ty)+\varphi(Tx)+\varphi(Ty))). \end{aligned}$$

Let $\{x_n\}$ be any sequence such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for any $n \in \mathbb{N}\cup\{0\}$. Then $x_n \geqslant x_{n+1}$ for any $n \in \mathbb{N}\cup\{0\}$. Therefore the sequence $\{x_n\}$ is a decreasing sequence and hence convergent. Let $\lim_{n\to\infty} x_n = z$. Since $[0,1]$ is complete we have $z \in [0,1]$. Therefore $z \geqslant Tz$ and hence $\alpha(z,Tz) \geqslant 1$ and $\mu(z,Tz) \leqslant 1$. We observe that $\alpha(x,Tx) \geqslant 1$ and $\mu(x,Tx) \leqslant 1$ for any $x \in X$. Hence all the hypotheses of Theorem 3.1 hold and $0 \in X$ is the unique fixed point of T with $\varphi(0) = 0$.

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